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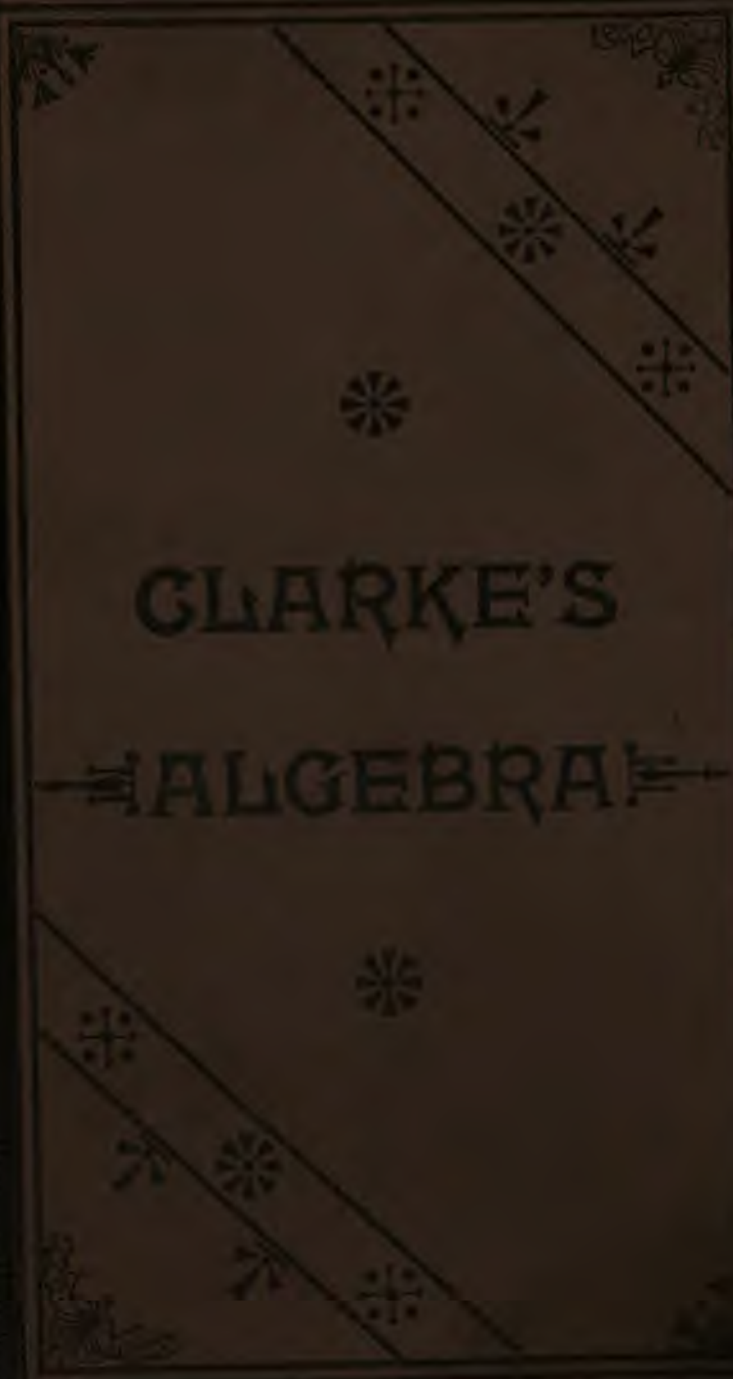
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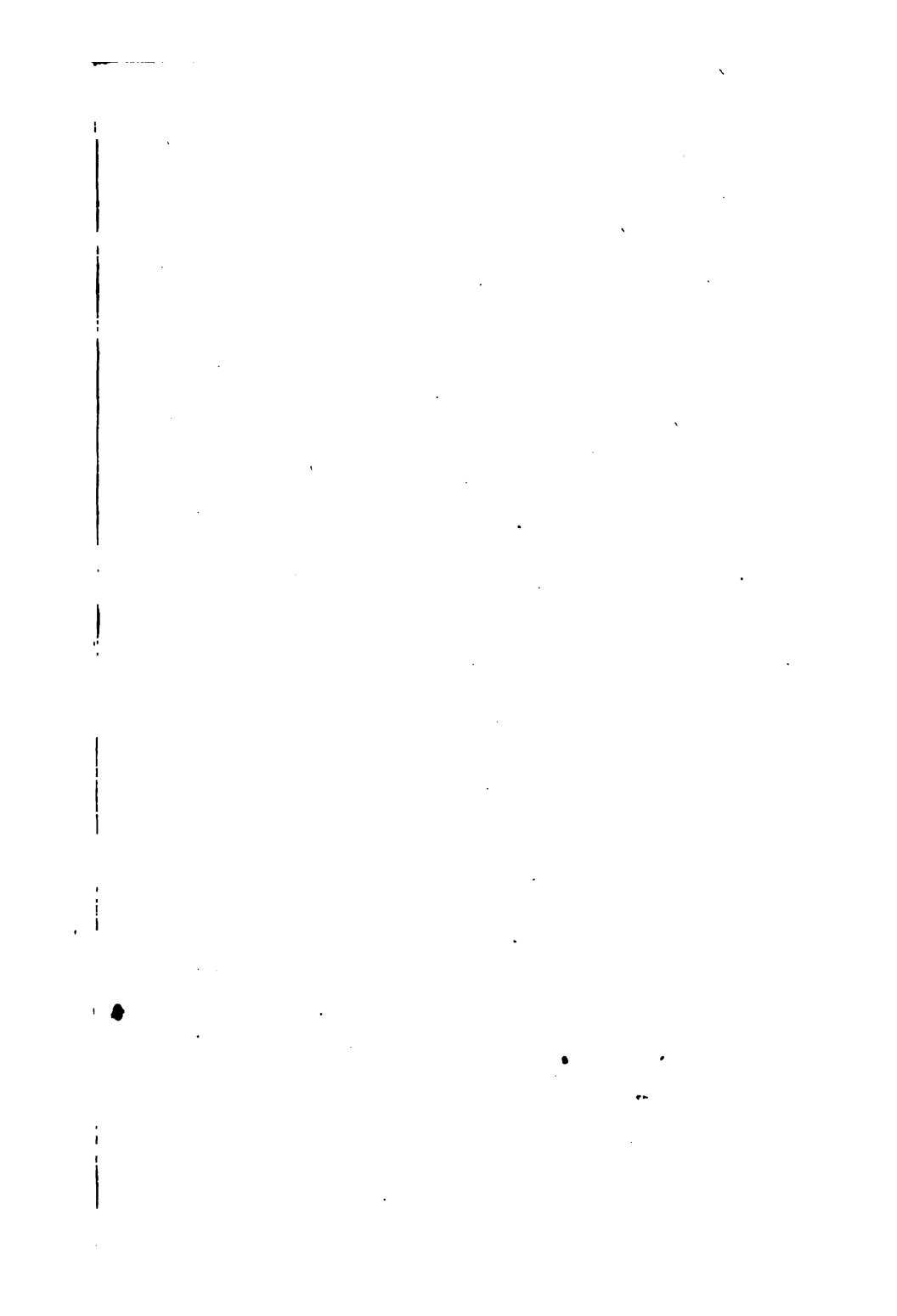
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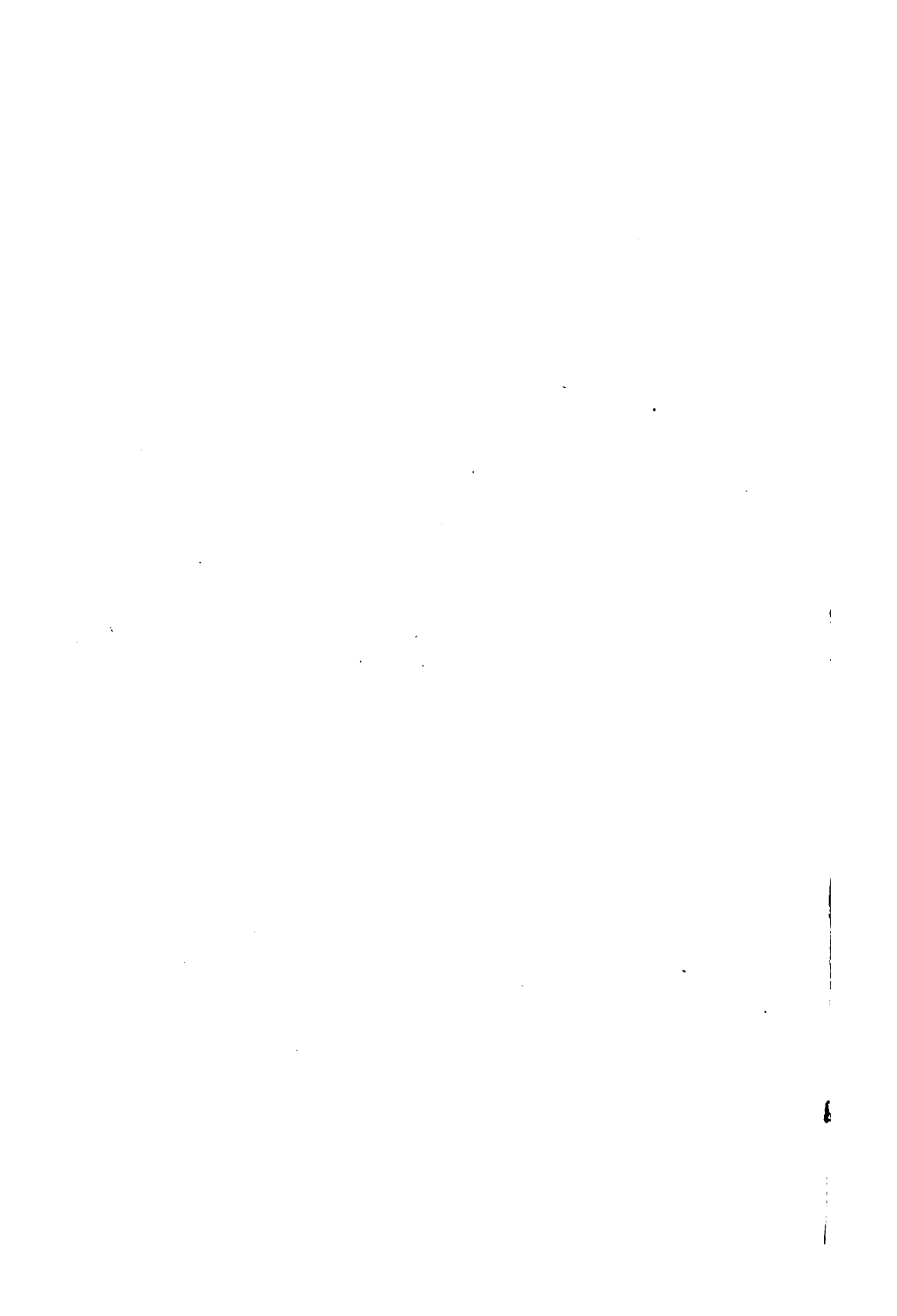


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FOR THE USE OF

HIGH SCHOOLS, ACADEMIES AND COLLEGES.

BY

JOHN BERNARD CLARKE,

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PREFACE.

THIS treatise is intended to serve as a text-book for those who desire to pursue a thorough course in mathematics.

Elementary work is usually undertaken either to complete a symmetrical culture-course, or to prepare the student for the Higher Mathematics, pure or applied. In the first case benefit is derived from the study simply because the exactness of statement, the rigorousness of demonstration, even of the most elementary principles, and the carefulness with which deductions are made therefrom, train the student to habits of accuracy in thought and closeness in reasoning. Again, the student seeking preparation for the higher mathematics is unfairly and untruthfully dealt with when important work, that may for the moment seem difficult or uninteresting, is neglected, and valuable time is inexcusably wasted, to secure a mere mechanical dexterity in the performance of operations that he will seldom, if ever, be called upon to perform in his advanced work. The most serious difficulties encountered in a college course arise from inadequate, inaccurate, and misdirected preparation. To assist in remedying the usual deficiencies in this regard is one aim of this work. Though intended more especially for those who propose to pursue mathematical studies of an advanced grade, it will, the author hopes, be equally helpful to those whose

mathematical work is merely a part of a symmetrical scheme of liberal education.

In this revised edition such changes have been made as have been suggested by class-room experience, or have been necessitated by the progress of science. The treatment of the Binomial Formula has been remodelled; the Theory of Logarithms is now so presented as to bring out all the most recent developments in this important department of the subject. The Theory of Determinants, the importance of which is now generally recognized, is discussed with sufficient detail to prepare the student for any of its ordinary applications.

The order of presentation of topics is that which seems to the author logically and practically best. Still the judicious teacher will adapt the work to the circumstances of his own pupils, not only in this regard, but also in the matter of distribution of time.

The author trusts that the work as now presented will prove acceptable alike to the energetic, progressive teacher, and the ambitious, earnest student.

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UNIVERSITY OF CALIFORNIA.
January, 1888.

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INTRODUCTION.

Science is knowledge, classified, systematized, arranged. The variety of objects of knowledge, of things known, renders necessary special subdivisions of Science. These subdivisions, each comprising all the systematized knowledge concerning one principal object, are called Sciences. Thus we have the systematized, classified, knowledge of Light, or the Science of Optics; the systematized knowledge of Animals, or the Science of Zoology; so with Botany, and the various other Sciences.

Mathematics is the classified, systematized knowledge of Quantity, under which general name we include all things that can be measured, and only these. Mathematics treats of the properties and relations of quantities. This science is distinguished from all others in that Mathematics alone comprises *absolute reasoned truth*. Mathematics is usually considered as either (1) Pure, or (2) Mixed, according as we discuss the properties and relations of quantity apart from any consideration of matter, or in direct connection with matter. Strictly speaking, all Mathematics is *Pure*, and what is called *Mixed Mathematics* is the mere *application* of the principles of the science. Nevertheless, this classification is convenient and useful, if we bear in mind its real nature.

Pure Mathematics, or Mathematics proper, is divided into Analysis, Arithmetic, and Geometry.

Analysis is that department of Mathematics in which the quantities considered are represented by letters and other symbols; and the relations subsisting among the quantities, as well as the operations to be performed upon them, are indicated by signs.

Arithmetic is a branch of Mathematics in which the quantities considered are represented by certain conventional symbols called figures, and the relations and operations discussed are indicated as in Analysis.

It is true that Arithmetical quantities, or numbers, are often represented by letters, as in Latin, Greek, and other ancient systems of notation. In these cases, however, the letters become, to all intents and purposes, *figures*, for each letter has a fixed and unchangeable signification; whereas in Analysis proper any letter may represent *any quantity*

whatever, and we may change the quantity represented by any particular letter at pleasure.

Geometry is the branch of Mathematics that treats of the properties and relations of certain magnitudes or quantities called the Geometrical Magnitudes, viz: Lines, Surfaces, Volumes and Angles. Geometry is sometimes combined, as it were, with Analysis, as in the Analytical Geometry and in Quaternions, which are, however, usually classed as subdivisions of Analysis.

Of the two remaining subdivisions of Analysis, Algebra, on which we are about to enter, is the one that treats of quantity as increasing and decreasing by finite differences. Calculus, on the other hand, treats of quantity as increasing and decreasing by infinitely small changes. Quantity as treated in Algebra is said to be Discontinuous while quantity as treated in Calculus is called Continuous. Hereafter the word *quantity* will be used to indicate *discontinuous quantity*.

Mixed Mathematics, or the application of the principles of Mathematics proper to the phenomena of the material universe, has various subdivisions, according to the phenomena selected, and the method of considering them. Such subdivisions are Physics, Mechanics, etc. Each of these subdivisions of Pure and of Mixed Mathematics is called a Science.

The importance of the Science of Algebra can hardly be overestimated. Sound and extended mathematical knowledge is absolutely impossible without a clear understanding and accurate knowledge of its principles and methods.

ALGEBRA.

CHAPTER I.

DEFINITIONS.

1. Quantity is anything that can be measured.

A thing is measured by finding how many times it contains something *of the same kind*, taken as a standard. This standard may be called the *unit* of measurement. Quantity is generally the answer to the question, "How much?" or "How many?" the answer being given in the form of a certain number of units. Quantity is, then, a unit or a collection of units. If the unit be 1, the quantity is said to be *Entire*, or *Integral*; if the unit be one of the equal parts into which 1 is divided, the quantity is said to be *Fractional*, or simply a *Fraction*. A quantity partly *Integral* and partly *Fractional* is called a *Mixed* quantity.

2. Mathematics is the science that treats of the properties and relations of quantities.

3. Algebra is a branch of Mathematics in which the quantities considered are represented by letters and other symbols, and the operations to be performed upon the quantities, as well as the relations of them, are indicated by signs.

4. The Quantities considered in Algebra are either (1) *Known*, or, (2) *Unknown*.

5. The Symbols used to represent quantities are:

1. *Letters*, as a, b, c, w, x, z ; the first or leading letters

of the alphabet (a, b, c, e , etc.) usually represent *known* quantities; the final letters usually represent *unknown* quantities, as x, y, z, w , etc. The letters may be written with indices, as x', b'', w''' , etc., read x *prime*, b *second*, w *third*, etc.; or with subscript figures, as m_1, c_2, r_3 , etc., read m *sub one*, c *sub two*, and so on. In using letters, and also other symbols, we sometimes speak of the letters or symbols themselves as quantities, but this is merely an abbreviated form of expression. Quantities represented by *letters* are frequently called *Literal* quantities.

2. *Figures*. These are used in the same general manner as in Arithmetic. Quantities represented by figures are called *Numerical*.

3. *The symbol zero* (0). This symbol is used either (1) as in Arithmetic to represent *no quantity*, or (2) to represent a quantity *less than any assignable quantity*.

4. *Infinity* (∞). This symbol is used to represent a quantity *greater than any assignable quantity*.

5. *Zero divided by zero* ($\frac{0}{0}$). This symbol, called the *symbol of indetermination*, is used to represent a quantity having under a particular supposition an infinite number of values. The symbols (3), (4) and (5) will be further explained hereafter.

Of these classes of symbols those under (1) are perfectly general in their application; the letters may represent any quantities whatever. The symbols under (2) are special; each symbol has but one meaning; represents but one quantity. The symbols (3), (4), (5), may each represent more than one quantity, but they are not so generally applicable as letters. The quantities represented by them must conform to the requirements laid down.

6. **The Operations** to be performed on quantities are indicated by the following signs:

(1.) The cross $+$, or *sign of Addition*. Placed between two quantities it indicates that their aggregate, or sum, is

to be taken. Placed before a quantity, it indicates that this quantity is to be added to some quantity, or that it is to be taken in an additive or positive sense. If no sign is written before a quantity) $+$ is always understood.

ILLUSTRATION. $m + n$, read m *plus* n , indicates that the aggregate, or sum, of m and n is to be taken; in other words, that n is to be added to m .

$+ r$, read *plus* r , indicates that r is to be taken in a positive or additive sense; or that r is to be added to some quantity not expressed. The same idea may be indicated by writing r simply, the sign $+$ being then understood.

7. If a quantity is to be taken several times in succession and added, we usually indicate the addition by writing the quantity once and prefixing to it a symbol indicating how many times it is to be taken and added. This symbol is called a *Coefficient*.

ILLUSTRATION. If m is to be taken 7 times in succession and added to any quantity, we indicate this addition by writing simply $+7m$, or $7m$. If m is to be taken 12 times in succession and added, we write $+12m$; if m is to be taken a times in succession and added, we write $+am$, or simply am .

8. (2.) The sign ($-$) *minus*, the *sign of Subtraction*. Placed between two quantities it indicates that the excess of the first over the second is to be taken, i. e., that the second is to be subtracted from the first; placed before a quantity it indicates that the quantity is to be taken in a *Negative* or *Subtractive* sense, the contrary of a *Positive* sense. Or, we may express the same idea by saying that the minus sign prefixed to a quantity indicates that the excess of some quantity over the given quantity is to be taken, i. e., that the given quantity is to be subtracted from some quantity.

ILLUSTRATION. $r-s$, read r *minus* s , indicates that the excess of r over s is to be taken: i. e., that s is to be subtracted from r .

$-x$, read *minus* x , indicates that x is to be taken in a negative sense.

It always means the precise opposite of $+x$. To illustrate, as x may represent any quantity whatever, let it represent a certain distance. Now, if $+x$ represents this distance when measured *upward*, $-x$ will represent the same distance measured *downward*. Let $+x$ represent a certain *gain*; then $-x$ represents a *loss* equal in amount to the gain x . If $+$ is prefixed to quantities indicating distance *North*, $-$ must be prefixed to those indicating distance measured *South*. If we suppose $+$ prefixed to a quantity to indicate that time is to be estimated *after* a certain date, $-$ prefixed to such quantities will indicate that the time is to be estimated *before* this date; if $+$ indicate *forward* measurement, $-$ indicates *backward* measurement, and so on. Whatever, in general, $+$ be assumed to indicate, $-$ will indicate the opposite.

9. If a quantity is to be taken several times in succession and subtracted from some quantity, we indicate this successive subtraction by writing the quantity (to be subtracted) but once, and placing before it a symbol indicating how often it is to be taken and subtracted. This symbol is always preceded by the minus sign, to indicate that the quantity is to be subtracted, and not added.

ILLUSTRATION. If we wish to indicate that m is to be subtracted eleven times in succession from some quantity, we write simply $-11 m$; if m is to be subtracted 15 times in succession, we write $-15 m$; if m is to be taken b times and subtracted from some quantity, we write $-bm$.

The symbol used to indicate this successive subtraction is called (as in the case of addition), a *Coefficient*. A *Coefficient* is positive or negative, according as it indicates that the quantity to which it is prefixed is to be taken several times additively or subtractively.

10. (3) The *Double sign* ($\pm$) or ($\mp$) read *plus or minus*, or *minus or plus*. Placed between two quantities it indicates that we are to take first their sum and then their difference, or the reverse, according as it is written $\pm$ or $\mp$. Placed before a quantity it indicates that the quantity is to be taken first additively, and then subtractively, or the reverse, according as it is written $\pm$ or $\mp$.

ILLUSTRATION. $a \pm b$ indicates that first the sum and then the difference of a and b is to be taken.

$\mp m$, read *minus or plus m* , indicates that m is to be taken first subtractively and then additively.

11. (4) The sign ($-$) of *Difference*. Placed between two quantities it indicates that the excess of one over the other, i. e., the *difference* between them, is to be taken, but does not indicate which is the greater.

ILLUSTRATION. $m - n$, read the *difference between m and n* , indicates that the excess of one over the other is to be taken, but gives no information as to the relative size of m and n .

12. (5) The sign ($\times$) of *Multiplication*, read *times* or *into* or *multiplied by*. Placed between two quantities it indicates that one is to be taken as many times as there are units in the other; i. e., that one is to be multiplied by the other. The quantity taken is called the *Multiplicand*, the number of times it is to be taken is indicated by the *Multiplier*. The result of taking the multiplicand the given number of times, (additively or subtractively, as the case may be), is called the *Product*. The *Multiplicand* and *Multiplier* are together called *Factors*; i. e., *makers*, of the *Product*. If a product is to be multiplied by another quantity, this product by another, and so on, the quantities multiplied together are all called *Factors*, and the result is called a *Continued Product*. In general, any product arising from multiplying together three or more quantities is a continued product.

When no misunderstanding is likely to occur, the multiplication of two or more quantities may be indicated by writing the factors in succession, with a point ($\cdot$) between them, or by writing them in succession with no intervening sign.

ILLUSTRATION. $m \times n$, $m.n$, and mn , all indicate the same fact, viz: m is to be taken additively as many times as there are units in n . If, however, we wish to indicate the multiplication of 7 by 8, we can not

write 7·8, as this would be mistaken for $7\frac{8}{10}$; while 78 would be read seventy-eight instead of seven times eight, or fifty-six. Again, we would not indicate the multiplication of a by $-b$ by writing simply $a-b$. If, however, we have a *continued product* of numerical factors, we may use the point, thus: 1·2·3·4·5·6·7. To indicate the multiplication of a by $-b$, it is best to write $a \times -b$, or $-b.a$, or $-ba$, or $a.(-b)$.

13 (6) *The sign of Division* ($\div$) read *divided by*. Placed between two quantities it indicates that the one on the left of the sign is to be divided by the one on the right. Thus $a \div b$, read *a divided by b*, indicates the division of a by b . The same operation may be indicated by the two dots alone, (as $x : y$, read *x divided by y*). It may also be indicated by the horizontal line alone, if the quantity to be divided be written above, and the other quantity below, this line, (as $\frac{x}{y}$, read as before). Still other methods of indicating division are by the use of the arc $\frown$, and the bar $|$ with a horizontal line attached, as $x \frown y$ and $y | x$, each read *x divided by y*.

The quantities operated upon in Division receive the names *Dividend* and *Divisor*, as in Arithmetic, and the result of the process is called the *Quotient*. The quotient is often called the *Ratio* of the dividend to the divisor; we shall meet this term again.

14. When all the factors of a product are equal, the product is said to be a *Power* of any one of the equal factors. The process of multiplying is then called *Involution*. If we have two equal factors the product is said to be the *second power*, or *square* of either; if we have four equal factors the product is the *fourth power*; if we have m equal factors the product is called the m^{th} power, and so on. The power is always named from the number of equal factors multiplied together to produce it. This *number of equal factors* is the *Degree* of the power.

ILLUSTRATION. $x x x x x$ is the *fifth power* of x , and this product or power is said to be of the *fifth degree*. Any one of the equal factors of a

power is called a *Root* of the power, and a root receives its *name* or *degree* from the number of times it is taken to form the power. If used *twice*, the root is the *square root*; if *three times*, the *cube root*; if *four times*, the *fourth root*; if *n times*, the *nth root*.

In the illustration just given x is the fifth root of $xxxxx$.

15. Instead of writing a factor the number of times it is to be taken to form a power, we generally write the factor but once, and write above, and a little to the right of it, a small symbol to indicate how often the factor is to be taken to form the power. This symbol is called the *Exponent of the power*; and sometimes, for brevity, the *Exponent of the quantity*.

This exponent is also called the *Exponential sign* or *sign of Involution*, the seventh of the signs of operation. It may be defined to be a small symbol written above and a little to the right of a quantity. It generally indicates the number of times the quantity is to be taken as a factor.

16. (8) The *sign of Evolution* $\sqrt{}$, or the *Radical* or *Root sign*. Placed before a quantity it indicates that the quantity is to be divided into two or more equal factors, one of which is to be taken; i. e., that some *root* of the quantity is to be taken. The number of equal factors required is indicated by a small symbol written in the opening of the sign. This symbol is called the *Index of the radical*, and fixes the *degree* of the latter.

ILLUSTRATION. $\sqrt[3]{x}$ indicate that x is to be divided into 3 equal factors; the expression is read the *cube* (or *third*) *root* of x . 3 is the *index of the radical*, which is of the *third degree*. So $\sqrt[n]{y}$ indicates that y is to be divided into n equal factors, i. e., the *nth root* is to be taken; n is the *index of the radical*, which is of the *nth degree*.

This sign is a modification of the initial r of the word *root*, and the index is the numeral formerly prefixed to r to indicate the root to be taken. As usually written $\sqrt{}$ the radical sign proper is combined with the *Vinculum*, a sign yet to be explained.

17. Evolution may also be indicated by the use of fractional exponents. Thus $\sqrt[n]{a}$ and $a^{\frac{1}{n}}$ mean the same. $\sqrt[n]{a}$

and $a^{\frac{1}{n}}$ are equivalent expressions for the n^{th} root of a . In general, to indicate any root of a quantity we write the quantity with an exponent equal to 1 divided by the index of the root. This method of indicating Evolution is very frequently and advantageously employed.

As the integral exponent indicates Involution (or raising a quantity to a given power), and the fractional exponent with the numerator 1 indicates Evolution (or resolving into equal factors), the combination of these two classes of exponents indicates both Involution and Evolution with respect to a quantity. In other words, if a quantity has an exponent of the form $\frac{m}{n}$, the quantity is to be raised to the m^{th} power, and the n^{th} root of the result is to be taken.

ILLUSTRATION. $a^{\frac{m}{n}}$ read a to the power m divided by n , indicates that a is to be raised to the m^{th} power, and this power is to be divided into n equal factors, one of which is to be taken.

The full discussion of exponents and indices, positive and negative, entire and fractional, will be given hereafter.

18. When several quantities are to be subjected to the same operation, certain signs, often called *Signs of Aggregation*, are employed. These are: the *Vinculum*, or horizontal bar —, the *Parenthesis* (), the *Brackets* [], the *Braces* { } and the vertical *Bar* |. Each indicates that the quantities comprehended are to be subjected to one operation, or that they are to be considered together, as one quantity.

ILLUSTRATION. $(a+b+c)[m+n]$ indicates that a , b and c are to be added, and their sum multiplied by the sum of m and n .

$x - (u+c)$ indicates that the sum of u and c is to be subtracted from x . The expression is read x minus the parenthesis $u+c$, or x minus the sum of u and c .

$\sqrt[3]{m+n}$, $\sqrt[3]{m+n}$, $\sqrt[3]{m+n}$ are equivalent expressions, each read the cube root of the quantity m plus n . The vinculum has been used so often in connection with the radical sign, that it has become almost a part of the latter, which is frequently written $\sqrt{\quad}$

19. The Relations of quantities to each other are also indicated by signs. The principal signs of Relation are:

(1) The sign of Equality ($=$). This is read *equals*. Placed between two quantities it indicates that they are equal.

An algebraic expression of the equality of two quantities, i. e., an expression consisting of two quantities connected by the sign of equality, is called an **Equation**. The quantity on the *left* of the sign is the *First Member*, and the one on the *right* the *Second Member*, of the *Equation*. Thus $ax = b$ is an *Equation*. ax is the *first member* and b is the *second*. So, also, $x + a = 2 + 3x$. Algebra, particularly Higher Algebra, is occupied almost entirely with the discussion of Equations.

20. (2) The sign of Inequality ($>$ or $<$). This sign, placed between two quantities indicates that the one toward which the opening of the sign is turned is the greater. If the opening is toward the left the sign is read *greater than*; if the opening is toward the right the sign is read *less than*. Thus $m < n$ is read *m is less than n*; $x > y$ is read *x is greater than y*. An expression consisting of two quantities connected by the sign of inequality is called an **Inequality** or **Inequation**. The quantity preceding the sign is the *First Member*; the quantity following the sign of inequality is the *Second Member*.

ILLUSTRATION. $x > y$, $m + 6b < 2 + 5c$, $3 < 5$, are *inequalities*. The *first members* are x , $m + 6b$, and 3 , respectively, and the *second members* are y , $2 + 5c$ and 5 , respectively.

21. (3) Signs of Ratio and Proportion. The quotient obtained by dividing a quantity by another of the same kind, is called the **Ratio** of the first to the second, or of the second to the first. A ratio may be indicated by any of the methods (13) of indicating division, but the sign most commonly used is the sign $:$, which is generally called the *sign of Ratio*.

ILLUSTRATION. The quantities ab and b being supposed of the same kind; the ratio of either to the other is $\frac{ab}{b}$ or $\frac{ab}{ab}$ or $ab : b$ or $b : ab$. When the quantities are separated by the sign $:$, the one preceding the sign is the *antecedent*, and the one following the sign the *consequent*.

A Proportion is an algebraic expression of the equality of two ratios. Thus, if the ratio of a to b is equal to the ratio of c to d , we may express this fact by writing $\frac{a}{b} = \frac{c}{d}$ or $a : b = c : d$. The latter form of expression is the more common, the sign of equality being transformed to $::$. The combination $::$ forms the *sign of Proportion*; $:$ is read *is to*, and $::$ is read *as*.

ILLUSTRATION. $a : b :: c : d$ is read *a is to b as c is to d*. It indicates that the ratio of a to b is equal to the ratio of c to d . If more than two ratios are equal, the expression of their equality is called a *Continued Proportion*.

$a : b :: c : d :: e : f :: g : h$ is a continued proportion.

22. (4) The sign of Variation or General Proportion is the character $\propto$, read *varies as*. Placed between two quantities it indicates that as one increases or decreases the other increases or decreases in the same ratio. That is, if either changes, the other so changes that the ratio of the first to the second remains always the same.

ILLUSTRATION. $a \propto bx$, read *a varies as bx*, indicates that no matter how a and bx may increase or decrease, the ratio $\frac{a}{bx}$ remains the same.

23. The Symbols and Signs of Algebra make up what is called **Algebraic Language**. In this language, as in any other, there are certain *abbreviations*. The most common are as follows:

$\therefore$ means *therefore, consequently*, etc.

$\because$ means *since, because*, etc.

$1^o, 2^o, 3^o$, are the usual abbreviation for *First, Second, Third*, etc.

$\div$ is often used to indicate that quantities following it are in *Arithmetical Progression*, as $\div a, b, c$. This fact may also be indicated by writing the sign $\dots$ between each term and the succeeding one, thus: $a \dots b \dots c \dots m \dots n \dots$.

$\div$ is often used to indicate that the quantities following it are in *Geometrical Progression*, the quantities being separated from each other by the sign $:$, as $\div a : b : c \dots$.

A series of dots or dashes, thus $\dots$, is often used to indicate that a sum or product, or an operation, is to be continued. These dots or dashes are called *leaders*; they answer very nearly to the etc. of ordinary language.

Instead of writing each one of a series of factors, we sometimes write only the last, with a vertical and horizontal line to indicate that its predecessors are to be supplied. Thus $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8$ may be written $\lfloor 8$ (read factorial 8). This notation is undesirable, and not often used.

Q. E. D. is written at the end of a course of reasoning or proof. Its meaning is, *which was to be proved*.

24. Any expression written in Algebraic language is called an Algebraic Expression.

As the letters used in algebraic expressions may represent any quantities whatever, we may assign to them any values whatever. The result obtained by assigning to each of the letters of any given algebraic expression a fixed value in numbers, and performing the operations indicated, is called the *Numerical Value* of the expression.

ILLUSTRATION. $3a^2m$ is 294, if $a=7, m=2$.

$a^2b + 3a^2c$ is 160, if $a=2, b=5, c=10$.

$m^2n + 2bn + nm^2 + n^4$ is 159, if $m=2, n=3, b=7$.

25. A Term is one of the parts into which an algebraic expression is divided by the sign $+$ or the sign $-$, or both. Algebraic expressions are classified according to the number of terms they contain. An algebraic expression consisting of but one term, i. e., an expression which, as written,

is not separated into parts by the signs + or —, is called a *Monomial*, or often simply a *Term*.

An expression consisting of *two* terms is called a *Binomial*, or, if the first term be positive and the second negative, frequently a *Residual*.

An expression consisting of *three* terms is called a *Trinomial*; an expression of *four* terms is a *Quadrinomial*; an expression consisting of more than four terms is called a *Polynomial*.

The name *Polynomial* may also be applied to any algebraic expression of more than one term. The word *multinomial* is very often used to denote an algebraic expression of more than one term.

26. A Monomial, or term, consists in general of two parts, a *Coefficient* and a *literal part* or *Unit*.

The **Coefficient** is that factor of the term that shows how many times the other part (*i. e.*, the term exclusive of the coefficient) is to be taken additively or subtractively. It may be numerical or literal, or partly numerical and partly literal. If a term be divided into *any two factors* we may consider either one as showing how often the other is to be taken additively or subtractively, according as the given term is positive or negative. Hence either one of these factors may be considered the *coefficient*, the other being called the *unit*.

27. The Degree of a term is the *number of literal factors* in the term. As the exponent of each letter (if positive and integral) indicates the number of times the quantity represented by the letter is taken as a factor, the *Degree of a term* is indicated by the *sum of the exponents of the letters of the term*.

Literal factors are sometimes called *dimensions*; the degree of a term is then the *number of dimensions*.

If no exponent be written, the exponent 1 is understood, and if fractional exponents occur, it is usual to consider only the numerators.

Homogeneous terms are such as are of the same degree. If all the terms of a polynomial are homogeneous the polynomial is said to be homogeneous, and its degree is indicated by the sum of the exponents of the letters in any one of its terms. We do not usually speak of the degree of a polynomial that is not homogeneous.

ILLUSTRATION. $m^3x + 3x^4 + 7x^2y + x^2y^2$ is a homogeneous polynomial of the fourth degree.

28. An algebraic expression is said to be in its **Simplest Form** when it contains the fewest terms of the lowest degree possible.

29. Similar Terms are terms having the same letters, affected by the same exponents.

We sometimes treat as similar terms having the same *unit*, but such terms are not (strictly speaking) always similar.

Dissimilar Terms are such as have different letters or different exponents, or both.

ILLUSTRATION. $3x^2y$, $2x^2y$, $7x^2y$ and $20x^2y$ are similar.

$2x^2y$, $3x^2y$, $5xy^2$ and $3m^2n$ are dissimilar.

30. The truths of Algebra are embodied in Propositions, Rules and Formulas.

31. A Proposition is a statement, generally setting forth something to be proved or something to be done.

Propositions are classified as *Theorems*, *Axioms*, *Problems*, *Postulates*, *Corollaries* and *Scholiums*.

32. A Theorem is a statement (purporting to be a statement of fact), the truth or falsity of which is generally to be made evident by a course of reasoning, called a *Demonstration*. If the truth of the theorem is so evident that no demonstration is required, the theorem is called an **Axiom**.

33. A **Lemma** is a theorem proved merely for use in proving or solving some succeeding proposition.

34. A **Problem** is a proposition setting forth something to be done.

The process of doing what the problem requires is called the *Solution* or *solving* of the problem.

35. A **Postulate** is a problem, the solution of which is evidently possible.

36. A **Corollary** is a subordinate theorem, the truth of which has been made evident by some part of the demonstration of the main proposition.

37. A **Scholium** is a remark appended to a discussion, for the purpose of pointing out something important or peculiar, developed during the discussion.

38. A **Rule** is a formal statement of the method of performing some operation.

39. To **Deduce** a rule is to arrive at the rule as the result of a process of reasoning applied to the successive steps in the operation to be performed.

A rule may also be announced at once, and proved by a demonstration.

It is generally *illustrated* (not proved) by problems requiring us to perform simply the operations directed by the rule.

40. Problems of this character, in which the only or the principal thing to be done is to perform a particular operation as laid down in a rule, are called **Examples**.

41. A **Formula** is an algebraic expression of a rule or proposition.

CHAPTER II.

THE FUNDAMENTAL RULES.

SECTION I.

ADDITION.

42. **Addition** is the process of finding the simplest algebraic expression for the aggregate of two or more quantities.

The result of this process is called the **Sum**.

43. **To Deduce a Rule for the addition of**

I. *Monomials:*

1°. *When the terms are similar and have like signs.*

Let it be required to add $+3x$ and $+2x$. The positive signs indicate that the quantities to which they are prefixed are to be added to some quantity, and the coefficients indicate that x is to be added first three times in succession and then twice again in succession to this quantity. If we add x three times in succession to any quantity, we obtain that quantity increased by $x + x + x$, and if to this result x be added twice in succession, we obtain the quantity $+x + x + x + x + x$ or (7) the quantity increased by $5x$. In other words, the same result will be obtained by adding $5x$ at once to any quantity, as by adding first $3x$ and then $2x$ to this result. $5x$, then, is the simplest expression for the aggregate of $3x$ and $2x$; i. e., it is their *sum*.

Or, we may obtain this result by noting that if we add any three things whatever to two things of the same kind, we obtain five things of this kind. If each thing is called x , then we have $3x + 2x = 5x$.

Again, let it be required to add $-4y$ and $-2y$. $-4y$ indicates that $4y$ is to be subtracted from some quantity. $-2y$ indicates that $2y$ is to be subtracted from this quantity. In other words y is to be subtracted first 4 times in succe-

sion from some quantity, and then twice in succession from the result. If we subtract y four times in succession, and then twice from the result, we shall evidently obtain the same result as if we were to take $6y$ at once from the quantity. That is, the aggregate of $-4y$ and $-2y$ may be expressed $-6y$; or, the sum of $-4y$ and $-2y$ is $-6y$.

Or, we may arrive at the same result thus: A quantity preceded by the sign $-$ is to be taken in a sense opposite to that in which the same quantity is taken when preceded by the sign $+$. Hence $-4y$ and $-2y$ indicate that $4y$ and $2y$ are to be taken in a sense opposite to that in which $+4y$ and $+2y$ are taken. The aggregate, the result of putting together $-4y$ and $-2y$ must be contrary in sense to the aggregate of $+4y$ and $+2y$. The aggregate of these, as already explained, is $+6y$. Hence the aggregate (the result of putting together), of these quantities, taken in a contrary sense, is $6y$ taken in a contrary sense, i. e., is $-6y$. $-6y$ then, the simplest expression for the aggregate of $-4y$ and $-2y$, is the sum required.

As the letters here used may represent *any quantities whatever*, we deduce, for the addition of similar terms having like signs, the following

RULE.—Add the coefficients for a new coefficient, to which prefix the common sign and annex the common literal part.

EXAMPLES.

1.	2.	3.	4.	5.
$3xy$	$- 4xyz$	$13xz^{\frac{1}{2}}y^{\frac{1}{3}}$	$- 10ab^{\frac{1}{2}}c^{\frac{1}{6}}d^{\frac{1}{3}}$	$- 6mxy$
$5xy$	$- 3xyz$	$10xz^{\frac{1}{2}}y^{\frac{1}{3}}$	$- 5ab^{\frac{1}{2}}c^{\frac{1}{6}}d^{\frac{1}{3}}$	$- 7mxy$
$7xy$	$- xyz$	$12xz^{\frac{1}{2}}y^{\frac{1}{3}}$	$- 3ab^{\frac{1}{2}}c^{\frac{1}{6}}d^{\frac{1}{3}}$	$- 8mxy$
$2xy$	$- 11xyz$	$3xz^{\frac{1}{2}}y^{\frac{1}{3}}$	$- 17ab^{\frac{1}{2}}c^{\frac{1}{6}}d^{\frac{1}{3}}$	$- 9mxy$
$11xy$	$- 25xyz$	$17xz^{\frac{1}{2}}y^{\frac{1}{3}}$	$- 6ab^{\frac{1}{2}}c^{\frac{1}{6}}d^{\frac{1}{3}}$	$- 15mxy$
$28xy$	$- 44xy$	$55xz^{\frac{1}{2}}y^{\frac{1}{3}}$	$- 41ab^{\frac{1}{2}}c^{\frac{1}{6}}d^{\frac{1}{3}}$	$- 45mxy$

6. Add $-16mx^{\frac{1}{2}}y^{\frac{1}{2}}$, $-22mx^{\frac{1}{2}}y^{\frac{1}{2}}$, $-10mx^{\frac{1}{2}}y^{\frac{1}{2}}$, $-27mx^{\frac{1}{2}}y^{\frac{1}{2}}$, and $-mx^{\frac{1}{2}}y^{\frac{1}{2}}$.

7. Add $-5nx^2y^3$, $-12nx^2y^3$, $-15nx^2y^3$, $-22nx^2y^3$, and $-5nx^2y^3$.

8. Add $-13nxy^3$, $-5nxy^3$, $-7nxy^3$, $-2nxy^3$, and $-3nxy^3$.

2°. *When the terms are similar and have unlike signs.*

If we have given several similar terms having unlike signs, *i. e.*, some being positive and the others negative, we may add all the positive terms by the rule just given. We may add, also, the negative terms, by the same rule. Let $+5x$, then, represent the sum of all the positive terms in any example, and $-7x$ the sum of all the negative terms. We are required, then, to add $5x$ and $-7x$. Now, we may consider $-7x$ as the sum of $-5x$ and $-2x$ by (43 I, 1°). Hence, if to any quantity we add $5x$, and from the result take away $7x$, we shall obtain the same result as if we add $5x$, and then take away $5x$ and $2x$ in succession from the same. If, however, we add $5x$ to any quantity and then take away $5x$ from the sum, we shall leave the original quantity unchanged. If from this we now take $2x$, we shall have the quantity diminished by $2x$, or $-2x$. Hence we obtain the same result by taking $2x$ from a quantity as by adding $5x$ to the quantity and then taking away $7x$. In other words, $-2x$ means the same as $+5x$ + the term $-7x$; $5x + (-7x)$ is then expressed in the simplest way by $-2x$, or $-2x$ is the sum of $+5x$ and $-7x$. In a similar way we can prove that the sum of $-5x$ and $+7x$ is $+2x$. The reasoning being general, we deduce, for the addition of similar terms having unlike signs, the following

RULE.—*Find the sum of the positive coefficients; find the numerical sum of the negative coefficients. To the excess of one sum over the other prefix the sign of the greater and annex the common literal part.*

EXAMPLES.

(1.)	(2.)	(3.)	(4.)	(5.)
$-7mx^{\frac{2}{3}}$	$-5may^{\frac{1}{2}}$	$ab^{\frac{2}{3}}c^{\frac{1}{2}}$	$3a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$	$9amx^{\frac{2}{3}}$
$3mx^{\frac{2}{3}}$	$-13may^{\frac{1}{2}}$	$7ab^{\frac{2}{3}}c^{\frac{1}{2}}$	$-4a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$	$12amx^{\frac{2}{3}}$
$14mx^{\frac{2}{3}}$	$5may^{\frac{1}{2}}$	$13ab^{\frac{2}{3}}c^{\frac{1}{2}}$	$-7a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$	$-21amx^{\frac{2}{3}}$
$-22mx^{\frac{2}{3}}$	$17may^{\frac{1}{2}}$	$-26ab^{\frac{2}{3}}c^{\frac{1}{2}}$	$15a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$	$-17amx^{\frac{2}{3}}$
$-16mx^{\frac{2}{3}}$	$-2may^{\frac{1}{2}}$	$-5ab^{\frac{2}{3}}c^{\frac{1}{2}}$	$-5a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$	$15amx^{\frac{2}{3}}$
$18mx^{\frac{2}{3}}$	$-3may^{\frac{1}{2}}$	$11ab^{\frac{2}{3}}c^{\frac{1}{2}}$	$20a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$	$-amx^{\frac{2}{3}}$
$-10mx^{\frac{2}{3}}$	$-may^{\frac{1}{2}}$	$ab^{\frac{2}{3}}c^{\frac{1}{2}}$	$22a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$	$-3amx^{\frac{2}{3}}$

6. Add $5(x+y)$, $7(x+y)$, $-3(x+y)$, and $-12(x+y)$.

Ans. $-3(x+y)$.

7. Add $6(m+n)$, $-12(m+n)$, $15(m+n)$, and $-10(m+n)$.

8. Add $32ax^{\frac{2}{3}}y^{\frac{1}{2}}$, $-57ax^{\frac{2}{3}}y^{\frac{1}{2}}$, $-13ax^{\frac{2}{3}}y^{\frac{1}{2}}$, and $35ax^{\frac{2}{3}}y^{\frac{1}{2}}$.

9. Add $2a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$, $3a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$, $9a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$, and $-17a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}$.

10. Add $5a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{5}{6}}$, $-7a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{5}{6}}$, $-3a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{5}{6}}$, and $2a^{\frac{1}{2}}b^{\frac{2}{3}}c^{\frac{5}{6}}$.

3°. Dissimilar Terms.

As we cannot add things that are not of the same kind in such a way as to get an expression involving things of one kind only, we cannot add terms involving different literal parts so as to get one term. Thus $3x$ and $2y$ are dissimilar terms. The things $3x$ and $2y$ are not of the same kind; we cannot by adding y a certain number of times get a sum involving x alone, nor will the addition of x one or more times produce a result involving y alone. When we have dissimilar terms to add, then, we simply connect them by the sign $+$, observing that when $+$ immediately precedes — it is not necessary to express the $+$ sign. Hence the

RULE.—Connect by their signs the terms to be added.

EXAMPLES.

1. Add a , $-2b$, and $3c$. *Ans.* $a - 2b + 3c$.
2. Add x , $-3y$, $5x$, and $-7m$. *Ans.* $6x - 3y - 7m$.
3. Add $2x$, 3 , $-y$ and $5a$. *Ans.* $2x + 3 - y + 5a$.
4. Add $ma^{\frac{1}{2}}$, -7 , nb , and $c^{\frac{1}{2}}$. *Ans.* $ma^{\frac{1}{2}} - 7 + nb + c^{\frac{1}{2}}$.
5. Add $x^{\frac{2}{3}}$, $-17x^{\frac{1}{2}}$, -5 and $y^{\frac{1}{2}}$. *Ans.* $x^{\frac{2}{3}} - 17x^{\frac{1}{2}} - 5 + y^{\frac{1}{2}}$.

44. II. *Polynomials.*

The addition of several polynomials may be indicated by writing in succession the quantities to be added and connecting them by the sign of addition. The only simplification possible is, then, the union of similar terms by (43). Suppose $3a + 2b + x$, $4b + x - a$, $4b - x$, and $-b + c + 2x + 5a$, are the given polynomials. Indicating the addition, we have $3a + 2b + x + 4b + x - a + 4b - x - b + c + 2x + 5a$ (1). As the result will be the same in whatever order we unite the terms, we may add first the terms involving a , then those involving b , and so on, uniting the results (43, 3°). $3a + 5a - a = 7a$ (43 2°); $2b + 4b + 4b - b = 9b$; $x + x - x + 2x = 3x$; c , not being similar to any other term, cannot be united with any other. Hence the polynomial (1) in its simplest form is $7a + 9b + 3x + c$, which is, therefore, the sum of the given polynomials. If the similar terms had come together in sets the simplification of the polynomial (1) would have been more easily accomplished, as we would then have been saved the trouble of searching for the terms similar to any particular term. If, then, the polynomials were so arranged that each set of similar terms should stand in one column, the addition in order of the columns would have given the result more readily. Hence the

RULE.—*Arrange the quantities to be added so that each set of similar terms may stand in a column. Add each column separately and connect the results by their proper signs.*

EXAMPLES.

(1.)

$$\begin{array}{r}
 3a - 7b + 5x \\
 2a - 3b + x \\
 a - b + 7x \\
 -6a + 13b - 15x \\
 22a - 8b - 6x \\
 -20a + b + 9x \\
 \hline
 2a - 5b + x
 \end{array}$$

(2.)

$$\begin{array}{r}
 3(m+n) + 5(x+y)^{\frac{1}{2}} \\
 7(m+n) - 2(x+y)^{\frac{1}{2}} \\
 8(m+n) - 4(x+y)^{\frac{1}{2}} \\
 -20(m+n) - 5(x+y)^{\frac{1}{2}} \\
 -13(m+n) + 7(x+y)^{\frac{1}{2}} \\
 5(m+n) - 8(x+y)^{\frac{1}{2}} \\
 \hline
 -10(m+n) - 7(x+y)^{\frac{1}{2}}
 \end{array}$$

3. $a^2b^{\frac{1}{2}} - 7b^{\frac{1}{2}}c^4 - m$, $5a^2b^{\frac{1}{2}} - 2b^{\frac{1}{2}}c^4 - 7m$, $3a^2b^{\frac{1}{2}} + 8b^{\frac{1}{2}}c^4 + 9m$,
 $-7a^2b^{\frac{1}{2}} - 9b^{\frac{1}{2}}c^4 - 6m$, $2a^2b^{\frac{1}{2}} + 10b^{\frac{1}{2}}c^4 - 10m$, $5a^2b^{\frac{1}{2}} - 7b^{\frac{1}{2}}c^4 +$
 $17m + d$.

Ans. $9a^2b^{\frac{1}{2}} - 7b^{\frac{1}{2}}c^4 + 2m + d$.

4. Add $v + 3w - 6z$, $7v - 8w + 10z + 7$, $20 + 6w - 10z + 9$,
 $-7w + 6v - 3 + z$, $5 + 2z - 3v + w$, and $2 + w - v - z$.

5. Add $3a^{\frac{1}{2}} - 6b - c^{\frac{2}{3}}$, $5a^{\frac{1}{2}} - 12b + 7c^{\frac{2}{3}} - 1$, $2b - a^{\frac{1}{2}} - c^{\frac{2}{3}}$, and
 $-5a^{\frac{1}{2}} + 10b + 5c^{\frac{2}{3}} + 1$.

Ans. $2a^{\frac{1}{2}} - 6b + 10c^{\frac{2}{3}}$.

6. Add $2(x+y) - w + 6$, $3(x+y) + 7 + w$, $2w - 20 - 4(x+y)$,
 $25 - 2w$, and $6(x+y) - 10 - 3w$.

Ans. $7(x+y) - 3w + 8$.

7. Add $19x - 12$, $15x - 2$, $13x + 20$, and $-37x - 2$.

8. Add $2x + 5(m+n) - y^e$, $3x - 5y^e - 7(m+n)$, $2x - 2(m+n)$,
 $5y^e - 8x + 11(m+n)$, and $4(m+n) - x$.

Ans. $-2x + 11(m+n) - y^e$.

9. Add $5x^2y^2 - 6x^2y^2 - 12x^4y^4$, $2x^2y^2 + 10x^4y^4 - x^2y^2$, $-17x^2y^2 +$
 $5x^2y^2$, $2x^2y^2 - 10x^2y^2 + 4x^4y^4$, and $9x^2y^2 + 8x^2y^2 - 7x^4y^4$.

10. Add $ax^{\frac{2}{3}} - bx^{\frac{1}{2}} - cx + d$, $3ax^{\frac{2}{3}} - 7bx^{\frac{1}{2}} + 8cx - d$, $3d - 7cx$,
 $5d - 8ax^{\frac{2}{3}} - bx^{\frac{1}{2}}$, and $6ax^{\frac{2}{3}} + 12bx^{\frac{1}{2}} + 2d$.

Ans. $2ax^{\frac{2}{3}} + 3bx^{\frac{1}{2}} + 7d$.

11. Add $7+3x^2y^2w^2-8m$, $3-2m-x^2y^2w^2$, $m+3x^2y^2w^2$, $15m-8+6x^2y^2w^2$, and $-5x^2y^2w^2-6$. *Ans.* $-4+6x^2y^2w^2+6m$.

12. Add $ax^2y-b^2x^2+7rs$, $2ax^2y-3b^2x^2-9rs$, $-7ax^2y+5b^2x^2+16rs$, and $ax^2y-13rs-8b^2x^2$.

13. Add $18-y^2+v$, $19-y^2+m$, $3y^2-38-4v$, and $v-6$.

14. Add $y+a^2+b^2$, $2y-a^2-6b^2$, $3b^2-7y$, and $y-4a^2$.

15. Add $(x+y)^{\frac{2}{3}}-7m+9(r+s)^c$, $5(x+y)^{\frac{2}{3}}+m-8(r+s)^c$, $2m-4(x+y)^{\frac{2}{3}}$, and $6(r+s)^c+9m+7(x+y)^{\frac{2}{3}}$.

Ans. $9(x+y)^{\frac{2}{3}}+5m+7(r+s)^c$.

16. Add $3x^{\frac{2}{3}}-7y^2w+3mv+7$, $2y^2w-3mv-x^{\frac{2}{3}}$, $9-2y^2w$, $-16+8x^{\frac{2}{3}}-5mv-y^2w$, and $3mv-6+9x^{\frac{2}{3}}-y^2w$.

17. Add $13-2(x+y)^c-a$, $52-6(x+y)^c-7a$, $16+5(x+y)^c+12a$, $3a-90$, $5(x+y)^c-5a$, and $17-3(x+y)^c+3a$.

Ans. $8-(x+y)^c+5a$.

18. Add $(x+a)^{\frac{1}{3}}-(y+c)^{\frac{2}{3}}+rs$, $2(x+a)^{\frac{1}{3}}-7(y+c)^{\frac{2}{3}}-3rs$, $5rs-(y+c)^{\frac{2}{3}}$, $-5(x+a)^{\frac{1}{3}}+10(y+c)^{\frac{2}{3}}$, and $3(x+a)^{\frac{1}{3}}+5rs$.

Ans. $(x+a)^{\frac{1}{3}}+(y+c)^{\frac{2}{3}}+8rs$.

19. Add $2m^2a^2l+5cd-x$, $4cd-m^2a^2l+7x$, $2m^2a^2l-cd$, $5m^2a^2l-4x$, and $3m^2a^2l-8cd+3x$.

20. Add $5a^{\frac{1}{2}}-6xy^2-3(m+n)^{\frac{1}{4}}$, $2a^{\frac{1}{2}}-8xy^2-5(m+n)^{\frac{1}{4}}$, $6a^{\frac{1}{2}}+12xy^2$, $-20a^{\frac{1}{2}}-7xy^2+8(m+n)^{\frac{1}{4}}$, $3a^{\frac{1}{2}}+9xy^2-4(m+n)^{\frac{1}{4}}$, $8a^{\frac{1}{2}}-16xy^2$, and $-2a^{\frac{1}{2}}+13xy^2+6(m+n)^{\frac{1}{4}}$.

Ans. $2a^{\frac{1}{2}}-3xy^2+2(m+n)^{\frac{1}{4}}$.

45. If dissimilar terms have a common literal part, the other factors in each term may be regarded as the coefficients of the common part. Then as each coefficient

shows how many times the common literal part is to be taken in the corresponding term, the sum of the coefficients will evidently show how many times this common part is taken in all the terms. Hence the sum of the given terms may be indicated by writing the common part, and to it prefixing, in parenthesis, the sum of the coefficients. Thus, the sum of ax , bx , and cx , is $(a+b+c)x$.

EXAMPLES.

1. Add ax^2y^2 , bx^2y^2 , cx^2y^2 , and dx^2y^2 .

$$\text{Ans. } (a+b+c+d)x^2y^2.$$

2. Add $mx^{\frac{1}{2}}y^{\frac{2}{3}}$, $wx^{\frac{1}{2}}y^{\frac{2}{3}}$, $7x^{\frac{1}{2}}y^{\frac{2}{3}}$, and $ax^{\frac{1}{2}}y^{\frac{2}{3}}$.

$$\text{Ans. } (m+w+a+7)x^{\frac{1}{2}}y^{\frac{2}{3}}.$$

3. Add abc , mlc , xyz , and dc .

$$\text{Ans. } (ab+ml+xy+dc)c.$$

4. Add $3ab(x+y)^{\frac{2}{3}}$, $5df(x+y)^{\frac{2}{3}}$, $rc(x+y)^{\frac{2}{3}}$, and $s(x+y)^{\frac{2}{3}}$.

5. Add $3ab+2xy+d$, $3my-3b+cd$, $gb-fd$, $6ib-2ry-wd$, and $b-ld$.

$$\text{Ans. } (3a-2+g+6t)b+(2x+3m-2r)y+(1+c-f-w-l)d.$$

6. Add $abx-3cy+gm$, $2x-dy-m$, $ex+fy-lm$, $kx-2cy-hm$, and $rx+py-qm$.

7. Add ax^2+by+c and dx^2+ry+l .

8. Add x^2+xy+y^2 , $ax^2-axy+ay^2$, and $bx^2+bxy-by^2$.

$$\text{Ans. } (1+a+b)x^2+(1-a+b)xy+(1+a-b)y^2.$$

9. Add $(a+b)x+(c-d)y-xq^{\frac{1}{2}}$, $(a-b)x+(3c+2d)y+5xq^{\frac{1}{2}}$, $2bx+3dy-2xq^{\frac{1}{2}}$, and $-4xq^{\frac{1}{2}}-dy-3bx$.

$$\text{Ans. } (2a-b)x+(4c+3d)y-2xq^{\frac{1}{2}}.$$

10. Add $a(x-y)^{\frac{1}{2}}+bxy+c(a+x)^e$, $-bxy+(a+c)(a+x)^e$, $(a-y)^{\frac{1}{2}}$, and $2bxy+(a-1)(x-y)^{\frac{1}{2}}-a(a+x)^e$.

$$\text{Ans. } 2a(x-y)^{\frac{1}{2}}+2bxy+2c(a+x)^e.$$

46. The following illustration may render clearer the different facts that we have met with in discussing addition. Suppose that the quantities a , b , etc., represent distances along a line, as AB . Let us begin at C to measure these

A . . . y . . . C . . . x . . . B

distances. If, now, we assume all distances to the right to be positive, by Art. 8 all distances to the left will be negative. Suppose each of the subdivisions in the figure to be represented by a . Now let it be required to add $3a$ and $2a$. If from C we count a twice in succession, we get to x ; the distance from C to x is $2a$. Measuring from x the distance a three times successively, we arrive at B ; xB represents $3a$. Then the distance from C to B represents the sum of $2a$ and $3a$. But CB contains 5 subdivisions or $5a$; in other words, $3a+2a=5a$. In a similar way we may show, by measuring $3a$ and $2a$ successively to the left (8) that $-3a+(-2a)=-5a$. Suppose we wish to add $-3a$ and $+5a$. Measuring the $-3a$ to the left, we get Cy . Measuring $+5a$ from y to the right (8) we arrive at x ; the distance from C to an ending point (i. e., the sum of $-3a$ and $+5a$) is Cx or $+2a$, as we found before. We could evidently arrive at the same result by measuring $5a$ to the right, as to B , and then measuring back, or to the left, the spaces to x for $-3a$. In a similar way we may show that $-5a+3a=-2a$. Now we see that any one of these results might have been obtained by prefixing to the numerical difference of the coefficients the sign of the greater and affixing the common literal part, and measuring the result off to the right or left, according as the sign is positive or negative.

47. To apply this illustration to the case of dissimilar terms, it is necessary to consider the subdivisions of AB as each equal to the unit 1. We then measure off a , b , c , etc., of these units. The discussion would in other respects differ little from the foregoing.

SECTION II.

SUBTRACTION.

48. Subtraction is the process of finding a quantity, which, added to one of two given quantities, will produce the other.

The quantity to be found is called the **Difference** or **Remainder**. The first of the two given quantities is called the **Subtrahend**, the second is called the **Minuend**. The subtrahend is said to be *subtracted* from the minuend.

49. Lemma. *The subtraction of one quantity from another may be indicated by changing the sign of the subtrahend and adding it, with this changed sign, to the minuend.*

Let a and b represent any two quantities, then:

$$\left. \begin{array}{l} (1) \ a-(+b)=a-b. \\ (2) \ a-(-b)=a+b. \end{array} \right\}$$

(1) $a-(+b)=a-b$. The truth of this is plain, if we remember that the second member may be considered as the first, with the sign $+$ omitted before the b , and we know that when no sign is written, the sign $+$ is always understood.

(2) $a-(-b)=a+b$. Let the difference between a and $-b$, whatever this difference is, be represented by d . Then $a-(-b)=d$ (3). Now, if we add the same quantity to each of two equals, the results will be equal. Hence $a-(-b)+(-b)=d+(-b)$. By (43, 3°) $d+(-b)=d-b$. Hence $a-(-b)+(-b)=d-b$ (4.) If, however, we add and subtract the same quantity to and from any given quantity, the latter remains unchanged. Hence $a-(-b)+(-b)=a$ (5). And since things equal to the same thing are equal to each other, $a=d-b$ (6). If now we add b to each of the equals a and $d-b$, the results will be equal. Therefore $a+b=d-b+b$ (7). For a reason already given, $d-b+b=d$, whence $a+b=d$ (8). Referring to equation (3) we have $a-(-b)=d$. Since things equal to the same thing are equal to each other $a-(-b)=a+b$.

As a and b in this discussion represent any quantities whatever, we may always indicate the subtraction of one quantity from another by changing the sign of the subtrahend and annexing it with this changed sign to the minuend. Q. E. D.

50. This proposition is sometimes treated as a direct consequence of the fact that prefixing the minus sign to any quantity indicates that the quantity is to be taken in a sense contrary to that in which it was originally taken.

$+b$ indicates that b is to be taken in a positive or additive sense. Hence $-(+b)$ indicates that b is to be taken in a negative or subtractive sense, which we indicate more simply by writing $-b$.

$-b$ indicates that b is to be taken in a negative sense. Hence $-(-b)$ indicates that b is to be taken in a contrary or positive sense, which fact is expressed more simply by writing $+b$.

51. To Deduce a Rule for the subtraction of one monomial from another.

From Art. 48, we see that to subtract any monomial from another, we may change the sign of the subtrahend, and then add the result to the minuend. Now, if the minuend and subtrahend be similar, we may unite these similar terms by the rule for addition, thus reducing the *Difference* to one term. If, however, the minuend and subtrahend are not similar, no such reduction can be effected, and we have for the *Difference* or *Remainder*, the minuend, followed by the subtrahend with its sign changed.

Hence, to subtract one monomial from another, we have the following

RULE.—*Change the sign of the subtrahend or conceive it to be changed, and proceed as in addition.*

EXAMPLES.

(1.)	(2.)	(3.)	(4.)	(5.)
$3xyz$	$4ab^2c^{\frac{1}{2}}$	$-6abx^c$	$-12xyz^{\frac{1}{2}}$	$-42(m+n)^{\frac{1}{2}}$
$-7xyz$	$2ab^2c^{\frac{1}{2}}$	$-2abx^c$	$4xyz^{\frac{1}{2}}$	$-50(m+n)^{\frac{1}{2}}$
$10xyz$	$2ab^2c^{\frac{1}{2}}$	$-4abx^c$	$-16xyz^{\frac{1}{2}}$	$8(m+n)^{\frac{1}{2}}$

6. From $3ax^2$ take $-7ax^2$.
7. From $-8r^2s^2c$ take $-12r^2s^2c$.
8. From $-8abc^2m$ take $7abc^2m$.
9. From $4dmx^3$ take $-6dmx^3$.
10. From $7amy^{\frac{2}{3}}z^{\frac{5}{6}}$ take $3amy^{\frac{2}{3}}z^{\frac{5}{6}}$.

Test the correctness of the results by adding the remainder to the subtrahend; the result should of course equal the minuend. (48.)

52. To Deduce a Rule for the subtraction of one polynomial from another.

Let a represent the sum of the additive or positive terms of the given polynomial minuend, and $-b$ the sum of the negative terms. Then $a-b$ represents the minuend. Similarly let $c-d$ represent the subtrahend. We are required, then, to subtract $c-d$ from $a-b$. If from $a-b$ we subtract c , we obtain (49) $a-b-c$. We were not, however, required to subtract c from $a-b$, but c after it had been diminished by d . We have subtracted, then, a quantity larger by d than our given subtrahend. The subtrahend is, by definition, the quantity to which, if we add the difference, we shall obtain the minuend. The sum of the subtrahend and remainder is then always equal to the minuend. Our subtrahend, c , being larger by d than the given subtrahend, our remainder is smaller by d than the required remainder. For if the sum of two quantities is to remain unchanged, and we *add* any quantity to one of the two that are to be added, we must *subtract* the same quantity from the other of the two. So if we *decrease* either by any quantity, we must *increase* the other by the same quantity. Hence, as we have taken a subtrahend too small by d , we must add d to the remainder obtained, in order to get the true remainder. We therefore add d to $a-b-c$, thus obtaining $a-b-c+d$

as the true *difference*. This result might evidently have been obtained at once by changing the signs of the subtrahend, or conceiving them to be changed, and proceeding as in addition. Hence the

RULE.—*Change the signs of the subtrahend or conceive them to be changed, and proceed as in addition.*

EXAMPLES.

(1.)	(2.)	(3.)
$3a - 8b - 7c + d$	$3xy(a+b)^{\frac{1}{2}} - 7m$	$9x^{\frac{1}{2}} + 6z - 12ax^{\frac{1}{2}}$
$-2a + 3b - 4c + 3d$	$5xy(a+b)^{\frac{1}{2}} - 3m$	$3x^{\frac{1}{2}} - 12z - 13ax^{\frac{1}{2}}$
$5a - 11b - 3c - 2d$	$-2xy(a+b)^{\frac{1}{2}} - 4m$	$6x^{\frac{1}{2}} + 18z + ax^{\frac{1}{2}}$

4. From $(m^2 - n^2)^{\frac{1}{2}} + 4(a+b) - 3(c+d)^{\frac{1}{2}}$ take $3(a+b) - 2(m^2 - n^2)^{\frac{1}{2}} + 3(c+d)^{\frac{1}{2}}$. *Ans.* $3(m^2 - n^2)^{\frac{1}{2}} + (a+b) - 6(c+d)^{\frac{1}{2}}$.

5. From $a^2x - 3c^2y - dm + e$ take $8a^2x - c^2y + dm - e$.

6. From $rx^{\frac{1}{2}} - cy^{\frac{1}{2}} - dw^2 + g$ take $5rx^{\frac{1}{2}} - 6cy^{\frac{1}{2}} + dw^2 - 8g$.

7. From $x^2y - 3x^{\frac{1}{2}}y^{\frac{1}{2}} - 6ay$ take $3x^2y - 9x^{\frac{1}{2}}y^{\frac{1}{2}} - 27ay$.

8. From $7ax^2y - 3bmx^2 - dr$ take $6ax^2y - 4bmx^2 - 4dr$.

9. From $3(x+y)^{\frac{1}{2}} - 8(a+b)^m - 6ay^2 + g$ take $9(x+y)^{\frac{1}{2}} - 7(a+b)^m + 8g$. *Ans.* $-6(x+y)^{\frac{1}{2}} - (a+b)^m - 6ay^2 - 7g$.

10. From $5(x+y)^2 - 8mx^2y - gm$ take $3(x+y)^2 + 8mx^2y + 6gm$.

11. From $19ax^2 - 16wx + 3m - 2y$ take $17ax^2 - 15wx - 7m + 6y$. *Ans.* $2ax^2 - wx + 10m - 8y$.

12. From $ax^2c - bx^2d - gy$ take $15ax^2c - 3bx^2d + gy$.

Ans. $-14ax^2c + 2bx^2d - 2gy$.

13. From $x^{\frac{1}{2}} - 3x^2y + 3xy^2 - y^{\frac{1}{2}}$ take $2x^{\frac{1}{2}} + 6x^2y - 6xy^2 + y^{\frac{1}{2}}$.

Ans. $-x^{\frac{1}{2}} - 9x^2y + 9xy^2 - 2y^{\frac{1}{2}}$.

14. From $6(a+y)^2 - 2(d+x)^2 - 5a$ take $6a - 3(d+x)^2 - 7(a+y)^2$.
Ans. $13(a+y)^2 + (d+x)^2 - 11a$.

15. From $3mx^2 - 8gy^2 - pw^2 + 6a^2$ take $5mx^2 + 7gy^2 + pw^2 + 7a^2$.
Ans. $-2mx^2 - 15gy^2 - 2pw^2 - a^2$.

16. From $a^2 - b^2 - c^2 + 8$ take $-a^2 + b^2 + c^2 - 10$.
Ans. $2a^2 - 2b^2 - 2c^2 + 18$.

17. From $6x^m y^n z - 5x^a y^b - g + d$ take $7x^m y^n z - 6x^a y^b + 3y - 4d$.
Ans. $-x^m y^n z + x^a y^b - 4g + 5d$.

18. From $3(x+a)^m - 5(g+b)^n - (c+d)^{\frac{1}{2}}$ take $5(x+a)^m - 6(g+b)^n + (c+d)^{\frac{1}{2}}$.
Ans. $-2(x+a)^m + (g+b)^n - 2(c+d)^{\frac{1}{2}}$.

19. From $(a-b)xy - (p+q)(x+y)^{\frac{1}{2}} - mx^2$ take $-(3+m)x^2 - axy + (2p-3q)(x+y)^{\frac{1}{2}}$.
Ans. $(2a-b)xy - (3p-2q)(x+y)^{\frac{1}{2}} + 3x^2$.

20. From $2x(x+y)^{\frac{1}{2}} - 3axy + 2abc$ take $-17axy + 11abc - 7x(x+y)^{\frac{1}{2}} - m$.
Ans. $9x(x+y)^{\frac{1}{2}} + 14axy - 9abc + m$.

53. Use of Parenthesis, Brackets, etc.

We have found that when several quantities are to be subjected to one operation, they are included in a parenthesis, or some other sign of aggregation, to which the sign indicating the operation to be performed is prefixed or annexed. Hence a parenthesis preceded by the sign $-$ indicates that the polynomial inclosed is to be subtracted from some quantity. This subtraction is performed (52) by changing the signs of the subtrahend, and proceeding as in addition. Hence we may remove the parenthesis by performing the operation indicated, *i. e.*, by changing the signs of all the included terms. Conversely we may inclose several quantities in a parenthesis preceded by the sign $-$, if we previously change the sign of each term to be introduced.

54. Of course a parenthesis preceded by the sign + may be either introduced or withdrawn without changing the signs of any of the terms considered.

EXAMPLES.

$$1. \quad 3x-2y+4z+a-(2x-3y-6z+5a)=$$
$$3x-2y+4z+a-2x+3y+6z-5a=x+y+10z-4a.$$

2. $6ax - (4m + 2ax + y - 7) = ?$

3. $12ax - 6y + 2m - (6ax + 5y - 3m + 9) = ?$

4. $3a^{\frac{1}{2}}y + 2b^2m^2 - (5a^{\frac{1}{2}}y - 2n^3d + 8b^2m^2) = 2n^3d - 2a^{\frac{1}{2}}y - 6b^2m^2$.

5. $\frac{1}{2}(a+x-y) - \frac{1}{2}(a-b-c) = ?$

Introduce within a parenthesis the fourth, fifth, sixth and seventh terms of each of the following polynomials:

6. $3axy - 5ab + 7c - g - 3am - 8ck + 7 + 2x.$

7. $5x-6y-7z-8w-10m+7n-8b+a.$

8. $6rs - 8ab - 9cd + 4ac - 7xy + 8yz - 9w + v.$

9. $ax^{\frac{1}{2}} - bx^{\frac{1}{4}} + cy^{\frac{1}{3}} - dy^{\frac{1}{5}} - qx^m + vy^n - ax^3 + dy^3.$

10. $6x^{\frac{1}{2}} - 7x^{\frac{1}{3}} + 8x^{\frac{1}{4}} - 9y^{\frac{2}{3}} + 13ay^{\frac{2}{5}} - 7x^m g^n + 8ag^3.$

55. The rules of subtraction may be illustrated in a manner analogous to that employed for the illustration of the rules in addition. To take a single case. Let it be required to subtract $-3a$ from $2a$. Let the subdivisions of AB indicate, as before, units of length each equal to a . We wish

$\overline{A \quad . \quad . \quad . \quad . \quad o \quad . \quad 2 \quad . \quad . \quad 5 \quad . \quad B}$

to measure off the distance $2a - (-3a)$. Measuring off, as before, from o , the $2a$, we arrive at 2. Now we have already (§) seen that the minus sign, prefixed to a quantity, changes its sense. $-3a$ alone would indicate that the distance $3a$ is to be measured toward the left from o , the origin.

of distances. Hence $-3a$ taken with a negative sign, $-(-3a)$, indicates that the distance must be measured off in the opposite direction, i. e., to the *right*. Measuring off then from 2 toward the right, the distance $3a$, we arrive at 5, and the distance from 0 to 5 or $5a$ is the distance $2a - (-3a)$. The distance $5a$ is $2a + 3a$. Hence $2a - (-3a) = 2a + 3a = 5a$. In a similar manner we dispose of the other cases that may arise with *similar* terms.

If the terms are *dissimilar*, as a and b , consider each of the subdivisions of AB as equal to the unit 1, and suppose a or b of these units to be measured off. The case of dissimilar terms may then be disposed of precisely as we have treated the above.

56. From the preceding discussions, it will be seen that the sum of two algebraic quantities may be greater or less than either of the quantities; the difference between two algebraic quantities may be less or greater than either the minuend or the subtrahend. Addition, then, does not, as in Arithmetic, always imply increase, nor does Subtraction always imply diminution. It may frequently happen that an *algebraic sum of two quantities is a numerical difference, and that an algebraic difference is numerically a sum*.

57. We have noticed in the course of the preceding discussions that

$$1. a + (+b) = a + b.$$

$$2. a + (-b) = a - b.$$

$$3. a - (+b) = a - b.$$

$$4. a - (-b) = a + b.$$

The sign before the parenthesis is called the **Sign of Operation**; the sign within the parenthesis is called the **Sign of the Quantity**; the sign that precedes the quantity b in the second members of the equations is called the **Essential Sign**. From the remarks on the use of the parenthesis,

we see that the *essential sign* will be + if the *sign of operation* and the *sign of the quantity* are alike. If these two are unlike, the *essential sign* will be —. It should be remembered that a and b here may represent any quantities whatever. For example, if $a=0$, the equations become

$$\left\{ \begin{array}{l} +(+b)=+b. \quad (1) \\ +(-b)=-b. \quad (2) \\ -(+b)=-b. \quad (3) \\ -(-b)=+b. \quad (4) \end{array} \right.$$

SECTION III. MULTIPLICATION.

58. Multiplication is the process of taking one quantity as many times as there are units in another.

59. The first quantity is called the **Multiplicand** and the second the **Multiplier**. The multiplicand and multiplier are together called **Factors**, and the result of the process of multiplication is termed the **Product**.

60. Multiplication is a special form of Addition; it is the process of adding a quantity to itself a certain number of times.

Thus, if we are to multiply 4 by 3, we are to add 4 to itself twice, *i. e.*, we are to take 4 three times and add it. If a is to be multiplied by c , a is to be taken c times and added. If $-m$ is to be multiplied by x , $-m$ is to be taken x times and added. Again, if m is to be multiplied by $-w$, m is to be taken w times and subtracted. (8) Multiplying $-a$ by $-b$ is taking $-a$ $-b$ times and adding, *i. e.*, (8) taking it b times and subtracting.

61. Lemma. *The value of the product of two or more factors is the same, whatever the order of the factors.*

Let it be required to multiply x by y . As already stated,

this product is indicated by simply writing the letters in succession, thus xy . Now we wish to prove that the value of the product will not be changed by changing the order of the factors. xy means that y is taken x times and added, thus $y+y+y+y+y+y+\dots$, there being x terms. Now, if we take unity, or 1, from each of these terms, we shall take x units from the whole sum. If we take 2 units from each, we get $2x$ units from the sum. If 3 units be taken from each term, $3x$ units are taken from the whole sum; if 7 units be taken from each of the x terms, $7x$ units will be taken from the whole sum or x terms. Finally, if y units be taken from each of the x terms, yx units will be taken from the whole sum, or x terms. When we have taken away y units from each, there is nothing left; we have taken the whole quantity, or xy . We have already shown that the number of units taken is yx . Hence, as things equal to the same thing are equal to each other, $xy=yx$; i. e., the change in the order of the factors has no effect on the value of the product.

Again, let it be required to multiply s , t , and r together. The product indicated in the ordinary way is str . Now, $st=ts$, by the principle just established in the case of two factors. Hence $str=tsr$. As already shown, $sr=rs$. Hence $str=tsr=trs$. By continuing the process, we can show that $str=tsr=trs=rts=srt=rst$. The same general method may evidently be extended to the case of more than three factors.

If y were negative in the product of two factors, we should simply substitute $-y$ for y in the series $y+y+y+\dots$ to x terms, and then take -1 instead of $+1$, and so on. If x also were negative, $-y$ would have to be taken x times subtractively, giving $-(-y)-(-y)-(-y)\dots$ to x terms. We have already shown that $-(-y)=+y$. Hence $+y$ may be substituted for $-(-y)$ in every term, giving $+y+y+y+y+\dots$ to x terms, on which we may reason as before.

62. To Deduce a Rule for multiplying Monomials.

I. Law of Coefficients and Exponents.

Let it be required to multiply $7a^2bc$ by $8ab^3c^2$. This multiplication is indicated thus, $7a^2bc \times 8ab^3c^2$, or $7aabc \times 8abbbcc$, or, since we may (61) change the order of the factors at pleasure, $7 \times 8aaabbbbccc$, which, by the aid of exponents (15), we may express thus, $56a^3b^4c^3$. Reviewing the work, we see that this result might have been obtained directly by multiplying the numerical coefficients together for a new (numerical) coefficient, and after this writing all the letters found in the multiplicand and multiplier, giving to each letter an exponent equal to the sum of its exponents in the two factors.

The same law deduced here for numerical exponents will apply to the case of literal exponents. For suppose we are to multiply a^m by a^n . Indicating the product in algebraic language, we have $a^m a^n$. Now a^m is an abbreviation for $aaaaa \dots$, the a being repeated m times as a factor; a^n is an abbreviation for $aaaaa \dots$, the a being repeated n times as a factor. Hence the expression $a^m a^n$ is an abbreviation for an expression $aaaaa \dots$, in which a is used first m times in succession as a factor, and after this n times. If then we count the number of factors, we shall count from 1 to m and then on n more; i. e., from 1 to $m+n$ inclusive. In other words, if a is taken first m times as a factor, and then n times, it is taken altogether $m+n$ times; the product, then, of a^m by a^n is a taken $m+n$ times as a factor, or, indicating this by the use of exponents, a^{m+n} .

Now, let it be required to multiply $3a^m b^r$ by $2a^n b^s$. Indicating the product by the use of the multiplication sign, we have $3a^m b^r \times 2a^n b^s$, or (61) $3 \times 2 \times a^m b^r a^n b^s$, or (61) $6a^m a^n b^r b^s$, or, as just explained, $6a^{m+n} b^{r+s}$. Examining this product, we see that it conforms to the same law for coefficients and exponents as the product in which the exponents were numerical.

II. Law of Signs.

In the discussion I., both factors were assumed positive. We shall now discuss the various cases that may arise in the multiplication of positive and negative quantities, and of both:

1. Let it be required to multiply $+x$ by $+y$. This means that x is to be taken y times and added. x taken once additively is $+x$; x taken twice and added is $+2x$; x taken three times and added is $+3x$; x taken y times and added is $+yx$ or xy (61).

2. Let it be required to multiply $+x$ by $-y$. This means that $+x$ is to be taken y times and subtracted. $+x$ taken once and subtracted is $-x$ (49); $+x$ taken twice and subtracted is $-2x$; $+x$ taken seven times and subtracted is $-7x$; $+x$ taken y times and subtracted is $-yx$ or $-xy$.

3. Let it be required to multiply $-x$ by $+y$. This means that $-x$ is to be taken y times and added. $-x$ taken once and added is (43) $-x$; $-x$ taken three times and added is $-3x$ (43); $-x$ taken y times and added is $-yx$ or $-xy$.

4. Let it be required to multiply $-x$ by $-y$. This means that $-x$ is to be taken y times and subtracted. $-x$ taken once and subtracted (49) is $+x$; $-x$ taken twice and subtracted is $+2x$; $-x$ taken five times and subtracted is $+5x$; $-x$ taken y times and subtracted is $+yx$ or xy .

Reviewing our work, we see that if the quantities to be multiplied have like signs, the product is positive; if they have unlike signs the product is negative. This is usually expressed for brevity, in a **Law of Signs**, thus: *Like signs produce + and unlike signs produce —, in multiplication.*

The **Laws of Coefficients and Exponents** and of **Signs** give, for the multiplication of Monomials, the following

RULE.—*Multiply the coefficients together for a new coefficient; after this product write the letters entering the multiplicand and multiplier, giving to each an exponent equal to the sum of its exponents in the two factors.*

If the multiplicand and multiplier have like signs, no sign need be written before the product, as + will be understood. If the two factors have contrary signs, write the minus sign before the product.

EXAMPLES.

1. Multiply $15a^2b^3c^4$ by $12ab^4c^2$. *Ans.* $180a^3b^7c^6$.
2. Multiply $-3a^5$ by $5a^{10}x$. *Ans.* $-15a^{15}x$.
3. Multiply $9a^mb^n$ by $3a^2mb^2$. *Ans.* $27a^{2+m}b^{n+2}$.
4. Multiply $6xyz^3$ by $-5ryz^2$.
5. Multiply $5\frac{1}{2}x^2y^3z^4$ by $5\frac{1}{2}xyz^2$. *Ans.* $30\frac{1}{2}x^3y^4z^6$.
6. Multiply $-13ax^2y^3$ by $4ay^2x^3$.
7. Multiply $6(a+b)^2yz$ by $3(a+b)^4y^2z^2$. *Ans.* $18(a+b)^6y^3z^3$.
8. Multiply $-5(x+y)^mgf(nz+a)^c$ by $6(x+y)^bg^2f(nz+a)^d$.
Ans. $-30(x+y)^{m+b}(nz+a)^{c+d}g^3f^2$.
9. Multiply $-6(x+y)^a(w+v)^b(m+n)^2$ by $-8(x+y)^2(w+v)^2(m+n)^c$.
10. Multiply $-8x^2y^3(m+n)^a(v+s)^b$ by $7(v+s)^c(m+n)^dx^my^3$.
Ans. $-56x^{2+m}y^{3+y}(m+n)^{a+d}(v+s)^{b+c}$.

63. To Deduce a Rule for the multiplication of polynomials.

I. *Let the multiplicand be a polynomial and the multiplier a monomial.*

Suppose that m represents the sum of all the positive terms of the polynomial given, and $-n$ the sum of all the negative terms of this polynomial. Then $m-n$ will represent the given polynomial, whatever it be. Suppose that we are required to multiply this by any monomial whatever, which we represent by r . We are required to

take $m-n$, r times, and add. By the rule for addition the result would be obtained by writing the polynomial r times in succession, keeping the similar terms in columns and adding each column separately, and then uniting these results by their proper signs. If now we should treat $m-n$ in this way, m would occur r times in the first column; and $-n$ r times in the second; i. e., m would be taken r times and added, giving us (43, 62) mr ; then $-n$ would be taken r times and added, giving us (43, 62) $-nr$. The sum of the first column would then be mr and the sum of the second, $-nr$. Hence, by the rule for addition, $mr-nr$ would be the result obtained by taking $m-n$ r times and adding. This is precisely what we mean by saying that $m-n$ is to be multiplied by r . Hence the result obtained by multiplying $m-n$ by r is $mr-nr$. A review of our work shows that we may obtain this result at once by multiplying each term of the multiplicand by the multiplier, giving each partial product its proper sign as determined by the law of signs (62).

A numerical illustration may render the foregoing a little plainer. Suppose we are to multiply $m-n$ by 3. This means that $m-n$ is to be taken three times and added; i. e., we are to add $m-n$, $m-n$, and $m-n$. Writing the quantities to be added, and adding in accordance with the rule, we have $3m-3n$ as the result. $3m$ $m-n$
 $-3n$ is, then, the product of $m-n$ and 3. $m-n$
 We see at once that we could have obtained this result directly by multiplying $m-n$
 each term of $m-n$ by 3, and connecting $3m-3n$
 the results by their proper signs. A similar illustration may be applied to any case that may arise.

From the foregoing considerations we deduce, for multiplying a polynomial by a monomial, the following rule:

RULE.—*Multiply each term of the multiplicand by the multiplier, and connect the results by their proper signs, as determined by the law of signs in multiplication.*

EXAMPLES.

1. Multiply $a^4 - a^3b + ab^3 - b^4$ by $5a$.
2. Multiply $x^3 - 4x^2y + xy^2 - y^3$ by $3xy^2$.
3. Multiply $(x+y)^n - (a+b)^n - y$ by $5acx$.
4. Multiply $(x+y)^n - (x+y)^n$ by $3(x+y)$.
 Ans. $3(x+y)^{n+1} - 3(x+y)^{n+1}$.
5. Multiply $x^3 - 3x^2y - 3xy^2 - y^3$ by $7xy$.

64. II. Let both multiplicand and multiplier be polynomials.

For reasons previously given $m-n$ may represent any polynomial multiplicand, and $r-s$ any polynomial multiplier. Suppose, then, we are required to multiply $m-n$ by $r-s$. This means that $m-n$ is to be taken $r-s$ times and added; that it is to be taken and added a number of times equal to the number of units in the difference between s and r . If $m-n$ be taken r times and added, the result, by (63), is $mr-nr$. We were not, however, required to take the multiplicand r times, but $r-s$ times. We have taken it then, s times too often, *i. e.*, we have taken it as often as we were required to do so, and s times more, since r contains both $r-s$ and s . We shall obtain, then, precisely the result required by removing this surplus, that is, s times the multiplicand; s times $m-n$ by (63) is $ms-ns$, and subtracting this from the first result we obtain (52) $mr-nr-ms+ns$ as the true result of the whole operation.

On comparing this result with the given quantities we see that it could have been obtained at once by multiplying the multiplicand by each term of the multiplier in accordance with the rule of (63) and connecting the results by their proper signs, or, adding them.

A numerical illustration may render this discussion somewhat clearer. Suppose, then, that we multiply $m-n$ by $7-2$. If we multiply $m-n$ by 7, we get $7m-7n$. We were required, however, to take $m-n$, not 7 times, but $7-2$ or 5 times. We have taken it twice too often, then, as 7 contains both $7-2$, or 5, and 2; and to get the correct result we must diminish $7m-7n$ by twice $m-n$ or by $2m-2n$ (63). Doing so, we obtain $5m-5n$, which is evidently the result obtained by taking $m-n$ $7-2$ or 5 times.

Any other case of this multiplication may be similarly illustrated.

From the foregoing considerations we deduce, for multiplying a polynomial by a polynomial, the following

RULE.—*Multiply the multiplicand by each term of the multiplier successively, and add the results.*

65. In adding the partial products it is convenient to have all the similar terms in columns, as we are thus enabled to add without any rearrangement of the terms. This is effected by arranging the terms of the factors according to the ascending or descending powers of some common letter, which is then called the *leading letter*.

It will be seen that as terms, to be similar, must have the same letters, affected with the same exponents, and as the exponents of the leading letter diminish (or increase), from left to right, no term can be similar to one that precedes or succeeds it in the same horizontal line. And, as the powers of the leading letter in the factors diminish (or increase) toward the right, the first terms of the successive partial products will be *lower* (or *higher*) with respect to the leading letter, than the first terms of their predecessors. They will therefore fall in columns with subsequent terms (*i. e.*, terms farther to the right) of these preceding products, if they are similar to any such terms.

This arrangement of the polynomials is one merely of convenience, and is not essential to the correctness of the work.

EXAMPLES.

1. Multiply
- $7x+3y+6z$

by $2x+y+2z$

$$\begin{array}{r}
 14x^2+6xy+12xz \\
 7xy \qquad +3y^2+6yz \\
 \quad +14xz \quad +6yz+12z^2 \\
 \hline
 14x^2+13xy+26xz+3y^2+12yz+12z^2
 \end{array}$$

2. Multiply
- $x^4+x^3y+x^2y^2+xy^3+y^4$

by $x-y$

$$\begin{array}{r}
 x^5+x^4y+x^3y^2+x^2y^3+xy^4 \\
 -x^4y-x^3y^2-x^2y^3-xy^4-y^5 \\
 \hline
 x^5 \qquad \qquad \qquad -y^5
 \end{array}$$

3. Multiply
- x^2+xy+y^2
- by
- x^2-xy-y^2
- .

4. Multiply
- $x^2+y^2+z^2-xy-xz-yz$
- by
- $x+y+z$
- .

$$\text{Ans. } x^3+y^3+z^3-3xyz.$$

5. Multiply
- $x^4+2x^3+3x^2+2x+1$
- by
- x^2-2x+1
- .

$$\text{Ans. } x^6-2x^3+1.$$

6. Multiply
- $y^7+y^6z+y^5z^2+y^4z^3+y^3z^4+y^2z^5+yz^6+z^7$
- by
- $y-z$
- .

$$\text{Ans. } y^8-z^8.$$

7. Multiply
- $3x^2-9xy+7xy^2+y^3$
- by
- $6ax+2y$
- .

8. Multiply
- $2x^3+5xy^2-2y^3$
- by
- $2x^2-5xy^2+2y^3$
- .

$$\text{Ans. } 4x^5-25x^2y^4+20xy^5-4y^6.$$

9. Multiply
- $x^5-x^4y+x^3y^2+x^2y^3+xy^4+y^5$
- by
- $x-y$
- .

$$\text{Ans. } x^6-y^6.$$

10. Multiply
- $x^4-x^3y+x^2y^2-xy^3+y^4$
- by
- $x+y$
- .

$$\text{Ans. } x^5+y^5.$$

11. Multiply
- $x^5+3x^2y+3xy^2+y^3$

by $x^2-3x^2y+3xy^2-y^3$

$$\begin{array}{r}
 x^7+3x^5y+3x^4y^2+x^3y^3 \\
 -3x^5y-9x^4y^2-9x^3y^3-3x^2y^4 \\
 \quad +3x^4y^2+9x^3y^3+9x^2y^4+3xy^5 \\
 \quad \quad -x^3y^3-3x^2y^4-3xy^5-y^6 \\
 \hline
 x^7 \qquad \quad -3x^4y^2 \qquad +3x^3y^4 \qquad -y^6
 \end{array}$$

12. Multiply $5ab-6a^2c-7a^3+3a^2b$ by $6a+2ac-3a^2$.

13. Multiply $3x^2+4xy-5y^2$ by $2x^2-6xy+4y^2$.

Ans. $6x^4-10x^2y-22x^2y^2+46xy^2-20y^4$.

14. Multiply $y^4-y^3+y^2-y+1$
by cy^2-by^2+ay

$$\begin{array}{r|l|l|l|l|l|l} cy^2-c & y^2+c & y^2-c & y^2+c & y^2-b & y^2+ay \\ -b & +b & -b & +b & -a & \\ & +a & -a & +a & & \end{array}$$

This product may also be written $cy^4-(c+b)y^3+(c+b+a)y^2-(c+b+a)y^4+(c+b+a)y^2-(b+a)y^2+ay$.

15. Multiply $x^4-x^3+x^2-x+1$ by $5x^2-6x^2+7x$.

The multiplication of polynomials is often indicated by writing them in succession, each inclosed in a parenthesis. When the multiplication thus indicated is performed, the expression is said to be **Expanded**.

16. Expand $(x-y)(x^4-x^3y+x^2y^2-xy^3+y^4)$.

17. Expand $(x+y)(x-y)(x^2+y^2)$. *Ans.* x^4-y^4 .

18. Expand $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$.
Ans. x^6-y^6 .

19. Expand $(x^2+y^2)(x^2+xy+y^2)(x^2-xy+y^2)$.

20. Expand $(5ab-6mc+d)(3ab+7c-8d)(c+d)$

PROPOSITIONS CONCERNING MULTIPLICATION.

66. Proposition I.—If no two terms of the product are similar the number of terms in the product will be equal to the number in the multiplicand multiplied by the number in the multiplier.

For, each term of the multiplier produces one term of the product for each term in the multiplicand. Hence each term of the multiplier produces as many terms in the product as there are terms in the multiplicand. The whole multiplier will therefore produce as many terms in the

product as there are units in the number of terms of the multiplicand multiplied by the number of terms of the multiplier.

67. Proposition II.—*The smallest number of terms that can ever occur in the product of two polynomials is two. These two terms are (1) the term arising from multiplying together the terms (of the polynomials) involving the HIGHEST powers of the leading letter; and (2) the term arising from multiplying together the terms containing the LOWEST powers of the same letter.*

For, these two terms will contain the leading letter, one to a higher and the other to a lower power, than any other term, since the exponent of the leading letter in any term of the product will be the sum of its exponents in those terms, of the original polynomials, that produced it. One term of those mentioned in the proposition will contain the sum of the *greatest* exponents coming together, and the other the sum of the *smallest* exponents. Hence these terms will not be similar to any other terms of the product, and will therefore not reduce with any of them.

68. Proposition III.—*The product of two polynomials will be homogeneous if the polynomials themselves are homogeneous, and the degree of the product will be expressed by the sum of the numbers indicating the degrees of the factors.*

For, the degree of any term of the product will be expressed by the sum of the numbers expressing the degrees of the terms producing it, since the exponent of any letter in any term of the product is the sum of the exponents of that letter in the terms producing such term. If all the terms of the multiplicand are of the same degree, and all terms of the multiplier are also of the same degree, the quantities added to get the degree of any term of the product will be the same as the quantities added to get the degree of any other term. All the terms of the product will therefore be of a degree expressed by the sum of the numbers expressing the degree of the two original polynomials.

69. Theorem.—*The square of the sum of two quantities is equal to the square of the first plus twice the product of the first and second, plus the square of the second.*

Let m and n represent any two quantities. Then $m+n$ represents the sum of these quantities. Multiplying $m+n$ by $m+n$ we obtain (64) $m^2 + 2mn + n^2$. This product is made up of the sum of the squares of the two given quantities, plus twice the product of the quantities themselves.

$$\begin{array}{r} m+n \\ m+n \\ \hline m^2 + mn \\ mn + n^2 \\ \hline m^2 + 2mn + n^2 \end{array}$$

70. Theorem.—*The square of the difference of two quantities is equal to the square of the first, minus twice the product of the first and second, plus the square of the second.*

Let m and n represent any two quantities; the difference between them is $m-n$. The product obtained (64) by multiplying $m-n$ by $m-n$ is $m^2 - 2mn + n^2$, which product is made up as stated in the proposition.

$$\begin{array}{r} m-n \\ m-n \\ \hline m^2 - mn \\ - mn + n^2 \\ \hline m^2 - 2mn + n^2 \end{array}$$

71. Theorem.—*The product of the sum and difference of two quantities is equal to the difference of their squares.*

If a and b represent the quantities, show by direct multiplication that $(a+b)(a-b) = a^2 - b^2$.

72. Theorem.—*Given any three quantities, the product of the first plus the second and the first plus the third, is equal to the square of the first, plus the sum of the second and third into the first, plus the product of the second and third.*

Let a , b , and c , be the quantities. Prove by direct multiplication that $(a+b)(a+c)$ is equal to $a^2 + (c+b)a + bc$.

Any one of the quantities may be taken for the first, or second, or third, and the form of the product will still be as stated in the theorem. Thus $(b+c)(b+a) = b^2 + (a+c)b + ac$.

73. Theorem.—*Given any three quantities, the product of the first minus the second, and the first minus the third, is equal*

to the square of the first, minus the product of the first into the sum of the second and third, plus the product of the second and third.

Prove by direct multiplication. Thus $(x-a)(x-b) = x^2 - (a+b)x + ab$.

74. Theorem.—Given any three quantities, the product of the first plus the second, and the first minus the third, is equal to the square of the first, plus the first into the second minus the third, minus the product of the second and third.

Prove by direct multiplication. Thus $(x+a)(x-b) = x^2 + x(a-b) - ab$.

75. Scholium.—The propositions (69, 70, 71) are evidently special cases of the theorems just given; cases in which the second and third quantities are equal.

76. Scholium.—Propositions 72, 73, 74, are all included in the following: THE PRODUCT OF ANY TWO BINOMIALS, HAVING THE FIRST TERMS THE SAME, EQUALS THE SQUARE OF THE FIRST TERM OF EITHER, PLUS THE FIRST TERM MULTIPLIED BY THE SUM OF THE SECOND TERMS, PLUS THE PRODUCT OF THE SECOND TERMS.

EXAMPLES.

Results to be written at once by aid of Arts. 69-76.

1. $(3b+2c)(3b+2c) = ?$ Ans. $9b^2+12bc+4c^2$.

2. $(7x+3y)^2 = ?$ Ans. $49x^2+42xy+9y^2$.

3. $(5ax+6by)^2 = ?$ Ans. $25a^2x^2+60abxy+36b^2y^2$.

4. $(9m^2x^3+5cdy^5)^2 = ?$
Ans. $81m^4x^6+90m^2cdx^3y^5+25c^2d^2y^{10}$.

5. $(3ca^2+6dm^3)^2 = ?$

6. $(9a-x)^2 = (9a-x)(9a-x) = ?$ Ans. $81a^2-18ax+x^2$.

7. $(3m-5y^3)^2 = ?$ Ans. $9m^2-30my^3+25y^6$.

8. $(6a-4b)^2 = ?$ Ans. $36a^2-48ab+16b^2$.

9. $(5cx^2y-2dm^3n)^2 = ?$
Ans. $25c^2x^4y^2-20cdm^3nx^2y+4d^2m^6n^2$.

10. $(7mxy-8wv^2)^2 = ?$

11. $(3a+6x)(3a-6x) = ?$ *Ans.* $9a^2-36x^2$.

12. $(5x^2y-7m)(5x^2y+7m) = ?$ *Ans.* $25x^4y^2-49m^2$.

13. $(6rx^2-8my^3)(6rx^2+8my^3) = ?$ *Ans.* $36r^2x^4-64m^2y^6$.

14. $(16mx^3-15sy^4)(16mx^3+15sy^4) = ?$
Ans. $256m^2x^6-225s^2y^8$.

15. $(17rsx^2-20y^6)(17rsx^2+20y^6) = ?$

16. $(x+7)(x+9) = ?$ *Ans.* $x^2+16x+63$.

17. $(5x+8)(5x+12) = ?$ *Ans.* $25x^2+100x+96$.

18. $(3y+6)(3y+4) = ?$ *Ans.* $9y^2+30y+24$.

19. $(7z+8)(7z+22) = ?$ *Ans.* $49z^2+210z+176$.

20. $(5x+4)(5x+10) = ?$

21. $(x-5)(x-12) = ?$ *Ans.* $x^2-17x+60$.

22. $(3x-7)(3x-9) = ?$ *Ans.* $9x^2-48x+63$.

23. $(5x-8)(5x-11) = ?$ *Ans.* $25x^2-95x+88$.

24. $(6y-12)(6y-13) = ?$ *Ans.* $36y^2-150y+156$.

25. $(9w-15)(9w-10) = ?$

26. $(x-7)(x+9) = ?$ *Ans.* $x^2+2x-63$.

27. $(3x-10)(3x+4) = ?$ *Ans.* $9x^2-18x-40$.

28. $(5x-7)(5x+8) = ?$ 32. $(5w-1)(5w+7) = ?$

29. $(7x-2)(7x+3) = ?$ 33. $(8w+10)(8w+11) = ?$

30. $(2x-5)(2x-11) = ?$ 34. $(9xy-7)(9xy+3) = ?$

31. $(4y-7)(4y-13) = ?$ 35. $(9y+3)(9y+5) = ?$

SECTION IV. DIVISION.

77. Division is the process of finding a quantity, which, multiplied by one of two given quantities, will produce the other.

The second of the two given quantities is said to be *divided* by the first, and is called the **Dividend**. The quantity by which it is divided is called the **Divisor**. The quantity sought is called the **Quotient**.

ILLUSTRATION.—If we are to divide $6x$ by 2, we are to find a quantity such that, on being multiplied by the divisor, 2, it will produce the dividend, $6x$.

78. As Multiplication is a special form of Addition, so Division is a special form of Subtraction. Thus, if we are to divide $12y$ by $3y$, we are to find how many times (in succession) $3y$ is to be subtracted from $12y$, that a remainder of 0 may be obtained. If from $12y$ we subtract $3y$, we obtain $9y$; if from this we subtract $3y$, we obtain $6y$; if from this we subtract $3y$, we obtain $3y$; and the subtraction of $3y$ from this result, leaves 0. $3y$ must then be subtracted four times (in succession) from $12y$, in order that a remainder of 0 may result. Since subtracting $3y$ four times leaves no remainder, we say that $3y$ is contained four times in $12y$. In other words, the quotient shows how many times the divisor is contained in the dividend, whence the name *quotient*. Division, then, is the process of finding how many times one quantity is contained in another. It is the converse of multiplication. In multiplication the factors are given and the product is required. The multiplier shows how many times the multiplicand is taken to form the product, i. e., how many times the product contains the multiplicand. In division, the product (or dividend) and one factor (the multiplicand) are given, and we are required to find the other factor (or mul-

tiplier). That is, we are required to find *how many times* the product or dividend contains the given factor or divisor.

79. To Deduce a Rule for the division of one monomial by another.

1°. Law of Coefficients.

Let it be required to divide $24a^3b^2c$ by $6ab$. We are required to find a quantity such that on being multiplied by $6ab$ it will produce $24a^3b^2c$. From the principles of multiplication we know that the coefficient of the quantity sought must be such as on being multiplied by 6 will produce 24. The only quantity that, multiplied by 6 produces 24 is 4. Hence the coefficient of the quotient is 4. In general,

The coefficient of the quotient equals the coefficient of the dividend divided by the coefficient of the divisor.

2°. Law of Exponents.

We know also from the principles of multiplication that if the quotient multiplied by $6ab$ produce $24a^3b^2c$, the letters a , b , and c must be found in the quotient, and the exponent of each must be such as added to the exponent of the same letter in the divisor, will give as a sum the exponent of this letter in $24a^3b^2c$. Hence the exponent of a must be 2, and of b , 1. Since the letter c enters the product, it must enter one or both of the factors of the product. It does not enter the divisor (or multiplicand). Hence it must enter the quotient (or multiplier). The literal part of the quotient is, then, a^2bc , and the coefficient 4; the quotient is therefore $4a^2bc$. In general,

The exponent of any letter in the quotient equals the exponent of that letter in the dividend, minus the exponent of the same letter in the divisor.

3°. Law of Signs.

Since the sign of the product is $+$ if the factors have

like signs, and — if they have unlike signs, when the dividend is positive the divisor and quotient have like signs, and when the dividend is negative the divisor and quotient have unlike signs. Thus, ab representing any dividend, and b any divisor:

$$\left. \begin{array}{l} +ab \div +b = +a \\ +ab \div -b = -a \\ -ab \div +b = -a \\ -ab \div -b = +a \end{array} \right\}$$

Hence we see that

If the dividend and divisor have like signs the quotient is +, and if they have unlike signs the quotient is —.

Since our reasoning is perfectly general, we deduce, for dividing one monomial by another the following

RULE.—*Divide the coefficient of the dividend by the coefficient of the divisor, for the coefficient of the quotient. After this write all the letters of the dividend, giving to each an exponent equal to the exponent of that letter in the dividend, minus the exponent of the same letter in the divisor.*

If the signs of the dividend and divisor are alike, prefix no sign to the quotient, the sign + being understood. If the signs of the dividend and divisor are unlike, prefix to the quotient the sign —.

EXAMPLES.

1. Divide $36a^2b^3c$ by $-2ac$. *Ans.* $-18ab^3$.
2. Divide $52x^2ys^2$ by $-4xs^2$. *Ans.* $-13xy$.
3. Divide $16xz^3$ by $4x$.
4. $-99m^2x^2yz + 11mys = ?$
5. $-105x^2ys^2 \div -7xs = ?$
6. $-86a^2(x+y)^2 \div 21\frac{1}{2}a(x+y) = ?$

7. $35abc^3d + 5ac = ?$

8. $153bx^3c + 51bc = ?$

9. $-912x^3y^2(m+n) \div 4xy = ?$

10. $-rx^2s^2y^2 \div -xzy^2 = ?$

11. $x^2y(x+y)^2 \div y(x+y) = ?$

12. $1728ax^3y(m+n)^3 \div -144xy(m+n) = ?$

Ans. $12ax(m+n)^2$.

80. Exact Division.—When the coefficient of the quotient is entire, or integral, and the exponent of every letter in the quotient is positive, the division is said to be **Exact**.

From (79) we see that the division of one monomial by another is exact if the coefficient of the dividend is exactly divisible by the coefficient of the divisor, and the exponent of each letter in the dividend exceeds the exponent of the same letter in the divisor.

81. Theorem.—*If, the divisor remaining unchanged, the dividend be multiplied (or divided) by any quantity, the quotient will be multiplied (or divided) by the same quantity.*

Let it be required to divide abc by a . The quotient (79) is bc . Multiplying abc by d we obtain $abcd$, which, divided by a , gives (79) bcd as a quotient. This quotient is d times the former quotient. Again, dividing abc by c we obtain ab , which, divided by a , gives b , a quotient equal to the first quotient, divided by c . As the letters used may represent any quantities whatever, we conclude that in any case, multiplying (or dividing) the dividend by any quantity, the divisor being unchanged, multiplies (or divides) the quotient by the same quantity.

82. Theorem.—*If, the dividend remaining unchanged, the divisor be multiplied (or divided) by any quantity, the value of the quotient will be divided (or multiplied) by the same quantity.*

Let it be required to divide $abcd$ by bc . The quotient is (79) ad . Multiplying bc by d we obtain bcd , which is con-

tained in $abcd$ only a times. The quotient a is equal to the old quotient divided by d .

Again, dividing the divisor by c we obtain b , which is contained in $abcd$ acd times. This quotient is equal to the first quotient, multiplied by c . As the letters employed may represent any quantities whatever, we conclude, etc., as stated in the theorem.

83. Theorem.—*If both dividend and divisor be multiplied or divided by the same quantity, the quotient will remain unchanged.*

This is a plain consequence of (81, 82).

83*. General Scholium.—A CHANGE IN THE DIVIDEND PRODUCES A LIKE CHANGE IN THE QUOTIENT; AND A CHANGE IN THE DIVISOR PRODUCES AN OPPOSITE CHANGE IN THE QUOTIENT.

84. Theorem.—*Any quantity affected with the exponent zero, is equal to unity.*

Let it be required to divide y^m by y^n . The quotient (79) is y^o . We know that any quantity is contained in itself once. Hence $y^m \div y^m = 1$. We have, then, $y^m \div y^n = y^o$ and $y^m \div y^m = 1$. Since things equal to the same thing are equal to each other, $y^o = 1$. Hence, as y may represent any quantity whatever, we conclude that any quantity affected with the exponent zero is equal to unity.

85. The Reciprocal of any quantity is unity (i. e., 1) divided by that quantity.

86. Theorem.—*Any quantity affected with a negative exponent equals the reciprocal of that quantity with a numerically equal positive exponent.*

Let it be required to divide y^m by y^{m+n} . The quotient (79) is y^{-n} . Now $y^m \div y^{m+n}$ may be written

$$\frac{y^m}{y^{m+n}} = \frac{y^m}{y^m y^n}$$

Dividing both dividend and divisor by y^m (83) we obtain

$$\frac{1}{y^n}; \text{ i. e., } \frac{y^m}{y^{m+n}} = \frac{1}{y^n}.$$

We have already found that

$$\frac{y^m}{y^{m+n}} \text{ equals } y^{-n}.$$

Since things equal to the same thing are equal to each other,

$$\frac{1}{y^n} = y^{-n}.$$

Since the letters employed may represent any quantities whatever, we conclude, etc., as stated in the theorem.

87. Theorem.—*Conversely; any quantity affected with a positive exponent equals the reciprocal of that quantity with a numerically equal negative exponent.*

For, letting y^m be any quantity with a positive exponent,

$$\frac{y^{m+n}}{y^n} = y^m.$$

Dividing both dividend and divisor by y^{m+n} , we have

$$\frac{y^{m+n}}{y^n} = \frac{1}{y^n}.$$

Since

$$\frac{1}{y^n} \text{ and } y^m \text{ are each equal to } \frac{y^{m+n}}{y^n},$$

they are equal to each other, or

$$y^m = \frac{1}{y^n}.$$

88. To Deduce a Rule for dividing a polynomial by a monomial.

From the principles of multiplication we know that since one of our factors is a monomial, and the product is a polynomial, the other factor must be a polynomial. That is, the dividend being polynomial and the divisor monomial, the quotient will be polynomial. From the manner of forming the product, we know that each term of the dividend has been obtained by multiplying the divisor by a cer-

tain term of the quotient. Hence, to divide a polynomial by a monomial, we must find a polynomial whose terms, being multiplied by the divisor, will produce the successive terms of the dividend. In other words, we must divide each term of the dividend by the divisor, connecting the results by their proper signs, as determined by the law of signs. An example may render this clearer.

Let it be required to divide $a^{m+n} - a^m b + a^m c^2$ by a^m . We are required to find a polynomial, such that its successive terms will, on being multiplied by a^m , produce the terms of the dividend. The first term, then, multiplied by a^m , must produce a^{m+n} . It must therefore (79) be a^n . For a similar reason the second and third terms must be $-b$ and $a^m c^2$, respectively.

Our reasoning being perfectly general, we deduce from the foregoing considerations the

RULE.—*Divide each term of the dividend by the divisor, connecting the results by their proper signs.*

EXAMPLES.

1. Divide $3x^2y^2z + 48x^2z^2y^2 + 27x^2y^2$ by $3x^2y^2$.

Ans. $z + 16z^2 + 9zy^2$.

2. Divide $35a^2b - 20abc - 15a^2b^2x + 5ab$ by $5ab$.

Ans. $7a - 4c - 3abx + 1$.

3. Divide $105x^2y^3 - 77y^2z + 14y^4z^2 - 84cxy^2$ by $7y^2$.

4. Divide $316m^2ny - 32mny^2 + 16m^2y^2$ by $4my$.

Ans. $79mn - 8ny + 4m^2y$.

5. Divide $72x^{m+n} - 36x^{m+1} - 52x^{m+1}y$ by $4x^m$.

Ans. $18x^n - 9x - 13x^1y$.

6. Divide $51a^2b^2c^2 - 34a^2c + 85ac^4$ by $17ac$.

7. Divide $63x^2y^2 - 108xy^2 + 99mx^2y^2$ by $9xy^2$.

8. Divide $57rs^2 - 93r^2s + 39r^4s^4$ by $3rs$.

9. Divide $9mx^2y - 18m^2x^4y^2 - 27m^3x^2y^2$ by $9mx^2y$.

10. Divide $117my^2z - 135mz + 144my^2$ by $9m$.

89. To Deduce a Rule for dividing a polynomial by a polynomial.

Let it be required to divide $3ab^2 + b^3 + a^2 + 3a^2b$ by $a + b$. This means that we are required to find a quantity such that, on being multiplied by $a + b$, it will produce $3ab^2 + b^3 + a^2 + 3a^2b$. The dividend, or product (64), is the sum of the *partial products* formed by multiplying the divisor by the different terms of the quotient. In adding, several similar terms may be combined so as to occur as but one term of the product. We know, however (67), that there are always *two* terms of the product that *cannot be reduced with any others*, viz: the terms containing the *highest* and *lowest powers*, respectively, of *any letter*. If, then, we take that term of the dividend containing the *highest power* of any letter, we know that it has been obtained *without reduction*, by multiplying the term of the divisor containing the highest power of this letter by the term of the quotient containing the highest power of the same letter. Hence, if we divide such a term of the dividend by that term of the divisor that assisted in producing it, we must obtain one term of the quotient. Now the term a^2 of our dividend contains a higher power of a than any other term. Hence, it must have been obtained *without reduction*, by multiplying the term of the divisor containing the highest power of a by the term of the quotient containing the highest power of a . Hence, if we divide a^2 by a , we shall obtain one term of the quotient. Dividing, we obtain a^2 . Now, since the dividend is the sum of all the partial products arising from multiplying $a + b$ by the successive terms of the quotient, if from the dividend we subtract one of these partial products, the result will be the sum of the

products of the divisor by the remaining terms of the quotient, and on this result we can reason precisely as on the original dividend. Subtracting, then, from $3ab^2 + b^3 + a^3 + 3a^2b$, the product $(a^2 + a^2b)$ of $a + b$ and a^2 , we obtain $3ab^2 + b^3 + 2a^2b$ as the *sum of the partial products arising from multiplying $a + b$ by the remaining terms of the quotient*. Since this is a product made up in the same manner as the original dividend, we can pick out the terms obtained without reduction, as in the case of the original dividend. $2a^2b$ then, was obtained *without reduction*, by multiplying the a of the divisor by that one of the remaining terms of the quotient that contains the highest power of a . Hence, this quotient term must be $2ab$, and subtracting $2a^2b + 2ab^2$ (for a reason already given), we have $ab^2 + b^3$, upon which we can reason precisely as upon the preceding remainder. Proceeding in the same manner we find b^2 as the next term of the quotient, and for a reason already given we subtract the product of $a + b$ and b^2 from the second remainder, thus obtaining a remainder of 0. We have now subtracted successively from our dividend the partial products produced by multiplying $a + b$ successively by a^2 , $2ab$ and b^2 . And since the remainder is 0, these must be all the partial products that were added together to produce the dividend; i. e., $a^3 + 2ab + b^3$ must be the entire quotient.

In the foregoing process we have taken first the term in which a is affected by the highest exponent, then the one in which the exponent of a is that next to the highest, and so on. Hence, if we had at once arranged the given polynomials according to the descending powers of a we should have avoided the *search* for these terms, as we should then have met them just in the order desired.

By reference to (67) it will be seen that we could have begun with the term involving the lowest power of a , and have proceeded regularly to the highest, reversing the order adopted in this discussion. We could also have reasoned on the powers of b just as we have reasoned on the powers of a .

From the foregoing considerations we deduce the following

RULE.—*Arrange both dividend and divisor according to the ascending or descending powers of some letter.*

Divide the first term of the dividend by the first term of the divisor for the first term of the quotient. Multiply the divisor by this term and subtract the product from the dividend.

Divide the first term of the remainder by the first term of the divisor, for the second term of the quotient. Multiply the divisor by this term of the quotient, and subtract the product from the first remainder.

Divide the first term of the second remainder by the first term of the divisor for the third term of the quotient. Multiply the divisor by the third term of the quotient and subtract the product from the second remainder.

Continue this process until a remainder of zero is obtained, or until a remainder is obtained the first term of which is of a lower degree with respect to the leading letter than the first term of the divisor.

90. If a remainder of zero is obtained the division is said to be **Exact**. When the division is not exact the remainder may be written at the right of the quotient over a horizontal line, below which is written the divisor.

91. It should be borne in mind that the *first term* of any remainder always means the *first term* of the remainder arranged in the same manner as the given polynomial.

EXAMPLES.

1. Divide $x^4 + y^4 - 4xy^3 + 6x^2y^2 - 4x^3y$ by $x^3 - 2xy + y^3$.

Arranging the polynomials according to the descending powers of x , we find the quotient of the first term of the dividend by the first term of the divisor to be x . Multiplying the divisor by this term and subtracting the result from the dividend, we find the first term of the remainder to be $-2x^2y$, whence the second term of the quotient is $-2xy$.

$$\begin{array}{r}
 x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \\
 x^4 - 2x^3y + x^2y^2 \\
 \hline
 -2x^3y + 5x^2y^2 - 4xy^3 \\
 -2x^3y + 4x^2y^2 - 2xy^3 \\
 \hline
 x^2y^2 - 2xy^3 + y^4 \\
 x^2y^2 - 2xy^3 + y^4 \\
 \hline
 \hline
 \end{array}
 \quad
 \begin{array}{r}
 x^2 - 2xy + y^2 \\
 x^2 - 2xy + y^2 \\
 \hline
 \hline
 \end{array}$$

Multiplying the divisor by this term, and subtracting the product from the first remainder, we find x^2y^2 as the first term in the second remainder, whence the next term of the quotient is y^2 , and multiplying the divisor by this, and subtracting, we find a remainder of 0. We consider only three terms of the first remainder, as the divisor contains only three terms. None of the partial products used can contain more than three terms.

2. Divide $x^4 + y^4$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$. *Ans.* $x + y$.

3. Divide $32x^5 - 243y^5$ by $16x^4 + 24x^3y + 36x^2y^2 + 54xy^3 + 81y^4$.
Ans. $2x - 3y$.

4. Divide $x^5 + 5x^3y + 5xy^3 + y^5$ by $x^3 + 4xy + y^3$. *Ans.* $x + y$.

5. Divide $a^6 - 3a^4b^2 + 3a^2b^4 - b^6$ by $a^3 - 3a^2b + 3ab^2 - b^3$.
Ans. $a^3 + 3a^2b + 3ab^2 + b^3$.

6. Divide $100a^5 - 440a^4k + 235a^3k^2 - 30a^2k^3$ by $5a^3 - 2a^2k$.
Ans. $20a^2 - 80ak + 15k^2$.

7. Divide $x^3 - y^3$ by $x - y$. *Ans.* $x^2 + xy + y^2$.

(8.)

$$\begin{array}{r}
 52m^5 - 93m^4p - 70m^3p^2 + 48m^2p^3 - 27mp^4 \\
 52m^5 - 28m^4p + 12m^3p^2 \\
 \hline
 -65m^4p - 82m^3p^2 + 48m^2p^3 \\
 -65m^4p + 35m^3p^2 - 15m^2p^3 \\
 \hline
 -117m^3p^2 + 63m^2p^3 - 27mp^4 \\
 -117m^3p^2 + 63m^2p^3 - 27mp^4 \\
 \hline
 \hline
 \end{array}
 \quad
 \begin{array}{r}
 13m^3 - 7m^2p + 3mp^2 \\
 4m^3 - 5m^2p - 9p^3 \\
 \hline
 \hline
 \end{array}$$

9. Divide $81x^4y^4 - 126abx^3y^3 + 49a^2b^2$ by $9x^2y^2 - 7ab$.

Ans. $9x^2y^2 - 7ab$.

10. Divide $8a^4 - 8a^3b^2 + 8a^2c^2 + 5b^4 - 3b^2c^2$ by $a^2 - b^2$.

Ans. $8a^2 - 5b^2 + 3c^2$.

11. Divide $6x^4 + 2y^4 - 9x^2y^2 + 4x^3y - 8xy^3$ by $2xy + 2x^2 - y^2$.

Ans. $3x^2 - xy - 2y^2$.

12. Divide $8x(y+2x)^2 - (y+2x)^3$ by $(y+2x)^2$.

Ans. $8x - y - 2x$.

13. Divide $5x^4 - 8x^3 + 9x^2 - 14x + 8$ by $x^2 - 2x + 1$.

14. Divide $x^4 + 3x^3 + x^2 - 3x - 2$ by $x^2 + 2x + 1$.

Ans. $x^2 + x - 2$.

15. Divide $2x^4 - 12x^3 + 19x^2 - 6x + 9$ by $x^2 - 6x + 9$.

Ans. $2x^2 + 1$.

16. Divide $x^7 + y^7$ by $x + y$.

Ans. $x^6 - x^5y + x^4y^2 - x^3y^3 + x^2y^4 - xy^5 + y^6$.

17. Divide $x^6 - y^6$ by $x - y$.

Ans. $x^5 + x^4y + x^3y^2 + x^2y^3 + xy^4 + y^5$.

18. Divide $x^5 - 2x^4 + 3x^3 - 7x^2 + 8x - 3$ by $x^2 - 3x + 3$.

Ans. $x^3 + x + 3$.

19. Divide $x^8 + y^8$ by $x + y$.

Ans. $x^7 - x^6y + x^5y^2 - x^4y^3 + x^3y^4 - x^2y^5 + xy^6 + y^7$.

20. Divide $a^5 - 5a^4x + 10a^3x^2 - 10a^2x^3 + 5ax^4 - x^5$ by $a^2 - 2ax + x^2$.

Ans. $a^3 - 3a^2x + 3ax^2 - x^3$.

PROPOSITIONS CONCERNING EXACT DIVISION OF POLYNOMIALS.

92. Proposition I.—*The exact division of one polynomial by another is impossible:*

1°. *When the term of the dividend that contains the highest power of any letter is not exactly divisible by the term of the divisor that contains the highest power of the same letter, or*

2°. *When the term of the dividend that contains the lowest power of any letter is not exactly divisible by the term of the divisor containing the lowest power of the same letter.*

For, the term of the dividend containing the highest power of any letter was obtained (66-7) *without reduction* by multiplying the term of the divisor containing the highest power of that letter by the term of the quotient containing the highest power of the same letter. Hence, the polynomials being arranged according to the powers of any letter, if there is no entire or integral quantity, that, multiplied by the first term of the divisor will produce the first term of the dividend, there can be no polynomial—with entire terms—that, multiplied by the divisor will produce the dividend. In other words, the exact division is impossible.

A similar course of reasoning will show the impossibility of exact division in the 2° case of the proposition.

93. Corollary I.—*The exact division of one polynomial by another is impossible if the divisor contain a letter not found in the dividend.*

The exponent of this letter in each term of the dividend is zero.

94. Corollary II.—*The exact division of a polynomial containing a certain letter by a second polynomial, not containing this letter is impossible, unless the coefficients of the different powers of this letter in the dividend are separately divisible by the divisor.*

For, we may consider the whole divisor as the coefficient of the zero power of the letter spoken of. The proposition then falls under (92).

95. Proposition II.—*The exact division of a monomial by a polynomial is impossible.*

For, if the division were possible, the quotient would be either a monomial or a polynomial. Now the product of a

monomial and a polynomial contains as many terms as the polynomial itself. Again, the product of a polynomial by a polynomial contains at least two terms (67).

96. Theorem.—*The difference of like powers of any two quantities is always exactly divisible by the difference of the quantities themselves.*

Let x and y represent any two quantities, and m any exponent. Then is $x^m - y^m$ always exactly divisible by $x - y$.

$$\begin{array}{r}
 x^m - y^m \quad | \quad x - y \\
 \hline
 x^m - x^{m-1}y \quad | \quad x^{m-1} + x^{m-2}y + x^{m-3}y^2 + \dots \\
 \hline
 x^{m-1}y - y^m \\
 x^{m-1}y - x^{m-2}y^2 \\
 \hline
 x^{m-2}y^2 - y^m \\
 x^{m-2}y^2 - x^{m-3}y^3 \\
 \hline
 x^{m-3}y^3 - y^m
 \end{array}$$

Dividing $x^m - y^m$ by $x - y$, we see that the last term of every remainder is $-y^m$, the same as the second term of the dividend. The exponent of x in the first term of each remainder is equal to the exponent of x in the dividend, minus the number indicating the order

of the remainder; while the exponent of y in this first term is the number indicating the order of the remainder; so that the sum of the exponents of x and y in the first term of any remainder found is always m .

In the course of the division the first term of each remainder is divided by x to obtain the next term of the quotient. Hence this term of the quotient is always the same as the first term of the remainder used, with the exponent of x in that term diminished by 1. While the term of this quotient obtained at any time contains x , the product of this term by y cannot be similar to y^m . Hence the product of this term of the quotient and y will be the first term of the next remainder. The first term of any remainder, then, will have the exponent of x less by 1, and the exponent of y greater by 1, than the first term of the next preceding remainder. In other words, the exponent of x diminishes by 1, and the exponent of y increases by 1, at each successive division. Hence, after five divisions the exponent

of x will be $m-5$; after m divisions it will be $m-m$ or 0. As the exponent of y increases by 1 at each successive division, after six divisions it will be y^6 , after m divisions it will be y^m . Hence, the first term of the m th remainder will be xy^m or $(84) y^m$; and as the second term, $-y^m$ is, by the nature of the division, the same in every remainder, the m th remainder will be y^m-y^m or 0. That is, after we have obtained m terms of the quotient, the remainder will be zero. Hence, the division of x^m-y^m by $x-y$ is exact.

87. Theorem.—*The sum of like powers of any two quantities is exactly divisible by the sum of the quantities themselves if the exponent of the power taken be odd, and not otherwise.*

Assuming x , y , and m as before, x^m+y^m will be exactly divisible by $x+y$ if m be odd, and then only.

Dividing x^m+y^m by $x+y$, the first and second remainders

$$\begin{array}{r|l}
 x^m+y^m & x+y \\
 x^m+x^{m-1}y & x^{m-1}-x^{m-2}y+x^{m-3}y^2-etc. \\
 \hline
 -x^{m-1}y+y^m & = -y(x^{m-1}-y^{m-1}) \\
 -x^{m-1}y-x^{m-2}y^2 & \\
 \hline
 x^{m-2}y^2+y^m=y^2(x^{m-2}+y^{m-2})
 \end{array}$$

are $-y(x^{m-1}-y^{m-1})$ and $y^2(x^{m-2}+y^{m-2})$, respectively. If, now, either of these remainders be exactly divisible by $x+y$, x^m+y^m will be exactly divisible by $x+y$. Take the second remainder. This will be exactly divisible by $x+y$ if $x^{m-2}+y^{m-2}$ be exactly divisible by $x+y$, and only then, (for y^2 is never exactly divisible by $x+y$). Hence,

If the sum of like powers of any two quantities be exactly divisible by the sum of the quantities, the sum of those like powers of the same quantities, of which the exponents are higher by 2 (than the exponents of the powers taken), will also be exactly divisible by the sum of the quantities themselves. And, if the sum of any like powers of two quantities is not exactly divisible by the sum of the quantities, the sum of the like powers of the second degree above those taken will not be

exactly divisible by the sum of the quantities. We know that $x+y$ is exactly divisible by $x+y$. Hence, x^3+y^3 is exactly divisible by $x+y$; and since x^3+y^3 is exactly divisible by $x+y$, x^5+y^5 is so divisible, and so on for all *odd* powers. Again, we know by actual trial that x^2+y^2 is not exactly divisible by $x+y$. Hence x^4+y^4 is not exactly divisible by $x+y$. Since x^4+y^4 is not exactly divisible by $x+y$, x^6+y^6 is not exactly divisible by $x+y$, and so on for all powers of an *even* degree.

98. Scholium.—This proposition may also be demonstrated in the same manner as the preceding, the m th remainder being zero if m is odd, and $2y^m$ if m is even.

99. Theorem.—*The difference of like powers of any two quantities is exactly divisible by the sum of the quantities themselves if the powers taken be of an even degree, and then only.*

We are required to show that x^m-y^m is exactly divisible by $x+y$ if m is even, and only then. Dividing x^m-y^m by

$$\begin{array}{r} x^m-y^m \quad | \quad x+y \\ x^m+x^{m-1}y \quad | \quad x^{m-1}-x^{m-2}y+x^{m-3}y^2- \\ \hline -x^{m-1}y-y^m = -y(x^{m-1}+y^{m-1}) \\ -x^{m-1}y-x^{m-2}y^2 \\ \hline x^{m-2}y^2-y^m \end{array}$$

$x+y$ we have a quotient, $x^{m-1}-x^{m-2}y+x^{m-3}y^2-\dots$, and the first remainder is $-y(x^{m-1}+y^{m-1})$. Now, if this remainder is exactly divisible by $x+y$, the original dividend will evidently be exactly divisible by $x+y$. Hence, if $x^{m-1}+y^{m-1}$ be exactly divisible by $x+y$, x^m-y^m will be exactly divisible by $x+y$. In other words,

If the sum of any like powers of any two quantities be exactly divisible by the sum of the quantities, then will the difference of the next higher powers be exactly divisible by the sum of the quantities themselves. From (97) we see that $x^{m-1}+y^{m-1}$ is exactly divisible by $x+y$ only when $m-1$ is odd. If, however, $m-1$ is odd, m is of necessity

even. Hence, in order that the first remainder may be divisible by $x+y$, m must be even. If, then, m is even, $x^m - y^m$ is exactly divisible by $x+y$. And as the dividend will not be exactly divisible by the divisor, unless the first remainder is exactly divisible by the divisor, $x^m - y^m$ will be divisible by $x+y$ only if m be even.

100. Theorem.—*The sum of like powers of any two quantities is never exactly divisible by the difference of the quantities themselves.*

The first remainder obtained in the process of dividing $x^m + y^m$ by $x-y$ is $y(x^{m-1} + y^{m-1})$. Unless this remainder be

$$\begin{array}{r}
 x^m + y^m \quad | \quad x-y \\
 x^m - x^{m-1}y \quad | \quad x^{m-1} + x^{m-2}y + \\
 \hline
 x^{m-1}y + y^m = y(x^{m-1} + y^{m-1}) \\
 x^{m-1}y - x^{m-2}y^2 \\
 \hline
 x^{m-2}y^2 + y^m
 \end{array}$$

exactly divisible by $x-y$, $x^m + y^m$ will not be exactly divisible by $x-y$. Hence,

The sum of like powers of any two quantities will not be exactly divisible by the difference of the quantities, unless the sum of the next lower powers of the same quantities is exactly divisible by the difference of the quantities. Now, $x+y$ is not exactly divisible by $x-y$. Hence, x^2+y^2 is not exactly divisible by $x-y$. Since x^2+y^2 is not exactly divisible by $x-y$, x^3+y^3 is not so divisible, and so on. Hence the sum of like powers of any two quantities is never exactly divisible by the difference of the quantities themselves.

CHAPTER III.

SECTION I.

FACTORING.

101. The **Factors** of a quantity are the quantities that, multiplied together, produce it.

102. **Factoring** is the process of finding two or more quantities such as, multiplied together, will produce a given quantity.

From the definition it is clear that a quantity is divisible by every one of its factors, and by the product of any two or more of them. If, then, one factor or more be known, we may always obtain the product of the others by dividing the quantity itself by the product of the given factors.

103. No general rule for factoring can be given. Skill in performing the process is attained by close observation and extensive practice, based on a thorough comprehension of the principles of multiplication and division. The accurate and intelligent use of the following propositions will, however, enable us to factor most of the quantities ordinarily met.

I. Monomials.

104. **Theorem.**—*The literal part of a monomial may be factored by writing the letters separately or in groups, giving to every letter such an exponent that the sum of all the exponents of the letter in these terms shall be equal to the exponent of the letter in the given monomial.*

This is a direct consequence of 62.

EXAMPLES.

1. Factor $93a^7b^2c$ and $60 a^{10}b^{12}c^3$.
2. Separate $36a^4b^5c^{2n+1}$ into two equal factors.
3. Separate $729x^3y^2(s+a)^{12}$ into three equal factors.
4. Factor $72a^3b^2$ and $120x^2y^3$.
5. Factor $108a^3b^2(c+d)^3$ and $54a^5x^4(c+d)^4$.

II. Binomials.

105. Theorem.—*If a binomial consists of the difference of the squares of two quantities, the factors of the binomial are the sum and the difference of the quantities themselves.*

See Art. 71.

106. Scholium.—Such a binomial may also be factored by 99.

107. Theorem.—*If a binomial consists of the difference of any two like powers of any two quantities, the difference of the quantities themselves is one factor of the given binomial.*

See Art. 96.

108. Theorem.—*If a binomial consist of the sum of any two like odd powers of any two quantities, or the difference of any two like even powers, the sum of the quantities themselves is one of the factors of the given binomial.*

See Arts. 97–99.

109. Scholium.—The forms of the quotients of Arts. 96–100 (and the *m*th remainders), should be carefully learned, so that they may be written at once, and thus the factors of a binomial be obtained without going through the division. It is easy to remember that if the divisor be the difference of the two quantities, the terms of the quotient are all

positive, and if the divisor be the sum of the two quantities, the odd-numbered terms of the quotient are positive, and the even-numbered terms are negative. The m th remainder is always either zero or of the form $\pm 2y^m$.

EXAMPLES.

1. Factor $x^2 - y^2$. *Ans.* $(x-y)(x+xy+y^2)$.
2. Factor $x^3 - y^3$.
3. Factor $32y^5 + z^5$.
Ans. $(2y+z)(16y^4 - 8y^3z + 4y^2z^2 - 2yz^3 + z^4)$.
4. Factor $64y^4 - b^4$.
Ans. $(2y+b)(2y-b)(4y^2+2by+b^2)(4y^2-2by+b^2)$.
5. Factor $81a^4 - 16c^4$.
6. Factor $1 \pm 27b^3$. *Ans.* $(1 \pm 3b)(1 \mp 3b + 9b^2)$.
7. Factor $243y^3 - 32c^3$.
Ans. $(3y-2c)(81y^2 + 54y^2c + 36y^2c^2 + 24yc^3 + 16c^4)$.
8. Factor $64x^5 - 1$.
9. Factor $x^7 \pm y^7$.
10. Factor $1 - 81y^4$. *Ans.* $(1-3y)(1+3y)(1+9y^2)$.

III. Trinomials.

110. Theorem.—If a trinomial consists of the sum of the squares of two quantities, PLUS twice the product of the quantities, the trinomial is the SQUARE OF THE SUM of these quantities.

This is a mere restatement of (69) in slightly different words.

111. Theorem.—If a trinomial is made up of the sum of the squares of two quantities MINUS twice the product of the quantities themselves, the trinomial is the SQUARE OF THE DIFFERENCE of these quantities.

Mere restatement of (70).

112. Scholium.—*If any trinomial of the form described in 110 or 111 be arranged according to the powers of either of the letters, the square of the middle term will be four times the product of the first and third terms.*

113. Theorem.—*If a trinomial is made up of the square of a quantity plus the first power of the quantity into the sum of a second and third, plus the product of the second and third, the factors of the trinomial are the first quantity plus the second, and the first quantity plus the third.*

See Art. 72.

114. Theorem.—*If a trinomial consists of the square of a quantity minus the first power of this quantity into the sum of a second and third, plus the product of the second and third, the factors of the trinomial are the first quantity minus the second, and the first minus the third.*

See Art. 73.

115. Theorem.—*If a trinomial consists of the square of a quantity, plus or minus the first power of this quantity into the difference between a second and third, minus the product of the second and third, the factors of the trinomial are, the first plus or minus the second, and the first minus or plus the third.*

See Art. 74.

EXAMPLES.

1. Factor $9x^2y^2 + 42a^2xy + 49a^4$.

Ans. $(3xy + 7a^2)(3xy + 7a^2)$.

2. Factor $25m^4c^2 + 80cm^2y^3 + 64y^6$.

3. Factor $m^6n^2 + 2m^4n^4 + m^2n^6$.

4. Factor $256x^4y^4 + 288mx^2y^2 + 81m^2$.

Ans. $(16x^2y^2 + 9m)(16x^2y^2 + 9m)$.

5. Factor $36x^4 + 144s^2x^2 + 144s^4$.

IV. Polynomials in general.

116. Theorem.—*If all the terms of a polynomial have a common factor, the polynomial may be divided by this factor, and the quotient and divisor are the factors of the given polynomial.*

Each term of the polynomial being exactly divisible by the common factor, the polynomial is exactly divisible by this common factor; and the quotient and divisor, multiplied together, produce the dividend.

ILL. The factors of $m^3 + 2m^2b + 3m^2c$ are, m^2 and $m + 2b + 3m^2c$.

117. Theorem.—*If a polynomial can be separated into parts, all having the same coefficient (whether this coefficient be monomial or polynomial) this coefficient is one factor of the polynomial. The sum of the parts, excluding the common coefficient, is the other.*

This is an obvious consequence of Art. 94.

117*. Corollary.—*If a polynomial can be separated into parts, the coefficients (monomial or polynomial) of these parts having a common factor, this factor is a factor of the polynomial. The other factor or factors may be obtained by dividing out this one.*

ILL. $12x^3 - 8xy - 9x^2y^2 + 6y^3 = (3x^2 - 2y)4x - (3x^2 - 2y)3y^2$. This equals $(3x^2 - 2y)(4x - 3y^2)$, since, if we divide the polynomial, as separated into parts, by $3x^2 - 2y$, the quotient is $4x - 3y^2$. Again: $(3x^2 - 2y)^2 4x - (3x^2 - 2y) 3y^2 = (3x^2 - 2y) [(3x^2 - 2y) 4x - 3y^2]$.

118. Scholium.—The separation of a polynomial into parts, which has just been illustrated, will be found a very good expedient. It frequently shows at once the presence of factors, otherwise difficult to detect.

EXAMPLES.

1. Factor $9mx^2 - 18mxy - 27m^2x^2z$.

Ans. $9mx(x - 2y - 3mx^2z)$.

2. Factor $5ab + 30a^2c - 15a^3d + 20a^4m$.

Ans. $5a(b + 6ac - 3a^2d + 4a^3m)$.

3. Factor $8a^3+16a^2b+8ab^2$. *Ans.* $8a(a+b)(a+b)$.
4. Factor $27m^3-108m^2x+270m^2x^2$.
5. Factor $17x^3-51x^2y+85xy^2-153x^2$.
6. Factor $ad+2ay+2by+bd=ad+bd+2ay+2by$.
 $= (a+b)d + (a+b)2y$.
 $= (a+b)(d+2y)$.
7. Factor $2x^3+3x^2+x$.
8. Factor $x^3+5x^2y+5xy^2+y^3$
 $= x^3+4x^2y+xy^2+x^2y+4xy^2+y^3$.
 $= (x^3+4xy+y^2)x + (x^2+4xy+y^2)y$.
 $= (x^2+4xy+y^2)(x+y)$.
9. Factor $6p^4-12pq^3-6p^3q+12q^4$. *Ans.* $6(p^2-2q^2)(p-q)$.
10. Factor $x^2y+ax+my+am$.
11. Factor $7x^2-24ay+21xy-8ax$. *Ans.* $(7x-8a)(3y+x)$.
12. Factor $ab^2d+xy+bdx+aby$. *Ans.* $(ab+x)(bd+y)$.
13. Factor $3x^4-25x^2y^2+28y^4$. *Ans.* $(3x^2-4y^2)(x^2-7y^2)$.
14. Factor $12a^2x^2+32abxy+21b^2y^2$.
 $\text{Ans. } (2ax+3by)(6ax+7by)$.
15. Factor $6m^3-13mx+7x^2$. *Ans.* $(6m-7x)(m-x)$.
16. Factor $acm+ag+bdm+bcm+adm+bg$.
17. Factor $rs^2+sx+xy+sy+rsy+y^2$.
 $\text{Ans. } (rs+x+y)(s+y)$.
18. Factor $x^3-xy^2-mx^2-ms^2+xz^2+my^2$.
 $\text{Ans. } (x^2-y^2+s^2)(x-m)$.
19. Factor $3x^4-14x^2y^2+8y^4-16y^2z^2+4x^2z^2$.
 $\text{Ans. } (3x^2-2y^2+4z^2)(x^2-4y^2)$.
20. Factor $60x^5-88x^3y-45x^4y^2+66x^2y^3+32xy^3-24y^4$.
 $\text{Ans. } (3x^2-2y)(5x^2-4y)(4x-3y^2)$.

SECTION II.

HIGHEST COMMON DIVISOR.

119. An Exact Divisor of a given quantity is a factor or divisor, such that the quotient obtained on dividing the given quantity by it is *entire* or *integral*. In speaking of *Divisors*, we generally mean *Exact Divisors*, and by *divisible* we usually mean *exactly divisible*.

ILLUSTRATION. $3, 7, m^2, x$ and y^4 are exact divisors of $21m^2xy^6$ and of $42m^2x^2y^4$.

120. A Common Divisor of two or more quantities is a factor or divisor that exactly divides each of them.

ILLUSTRATION. $7m^2x$ is a common divisor of $21m^2x$ and $35m^2x^2 - 42m^2x^3$.

121. The Greatest or Highest Common Divisor of two or more quantities is the Common Divisor that contains the *greatest coefficients* and the *greatest exponents*, of all the Common Divisors of the quantities.

The names *Greatest Common Divisor* and *Highest Common Divisor* are both somewhat faulty; the former in that it does not plainly involve the idea of *greatest exponents* or *highest powers*, the latter in that it does not plainly involve the idea of *greatest coefficients*.

ILLUSTRATION. $x^3 - y^3$ is *higher* in *degree* than $x^2 + xy + y^2$. Whether it is really *greater* or *less*, depends on the values assigned to x and y . Again, 8, the *cube* of 2, is *higher* in *degree* than 169, the *square* of 13, or 196, the *square* of 14.

We shall use the name *Highest Common Divisor* with the understanding that it involves the idea of *greatest coefficients*, as well as the idea of *greatest exponents*. We shall use the abbreviation C. D. for *Common Divisor*, and H. C. D. for *Highest Common Divisor*.

To Deduce a Rule for finding the Highest Common Divisor of two or more quantities.

I. Quantities readily factored.

122. From the definition of factors we know that any quantity may be exactly divided by its own factors, or the product of any two or more of them. For, such division would consist merely in striking out one, two, etc., factors, the quotient being the product of such as are not stricken out. Hence, if some factors are common to two or more quantities, we may divide each quantity by the product of two or more of these common factors, or by the product of all. From the definition of *degree* we know that of several terms that is highest in degree which contains the greatest number of literal factors. The greater the number and the value of common numerical factors entering a coefficient, the greater evidently will be the coefficient. Hence, the common divisor having the highest exponents, and the greatest coefficients will be the continued product of *all* the common factors, numerical and literal, of the given quantities. Hence, to find the highest common divisor of two or more quantities that can be readily factored we have the following

RULE.—*Separate each quantity into its simplest factors. Take the continued product of all such factors as are common to all the quantities.*

EXAMPLES.

Find the *highest common divisor* of:

1. $63a^2b^4c$ and $7ab^{10}d$.
2. $95a^{\frac{2}{3}}bc^2$ and $50a^{\frac{2}{3}}b^3c^{24}$.
3. $105m(x+y)^3$ and $245m(x+y)^5c$.
4. $18(x^4-y^4)$ and $27(x^2+y^2)$.
5. $93(x^5-y^5)$ and $62(x^2-y^2)$.

6. $98a^6(a^3-2ab+b^3)$ and $105a^3(a^3-b^3)$.

7. $26x^2(x^2+7x+12)$ and $13(x+3)$.

8. $yx^3-3x^2y-40xy$ and $(4x+32)y^2$.

9. $81(a^3-b^3)$ and $27(a^3-b^3)$.

10. $1-81y^4$ and $1+6y+9y^2$.

II. When the quantities cannot be readily factored.

123. Lemma.—*A Common Divisor of two or more quantities exactly divides the sum and also the difference of these quantities.*

This is an immediate consequence of (88). Or, let abc and acr be any two quantities, of which ac is a common divisor. Indicating their sum and difference, we have $acb \pm acr$. Dividing this by ac , we obtain (88) $b \pm r$; which is entire or integral, since b and r are both entire, i. e., the division is exact. Similar reasoning will evidently apply to the case of three or more quantities.

124.—Corollary.—*The highest common divisor of two or more quantities is an exact divisor of the sum and also of the difference of the quantities.*

125. Lemma.—*An exact divisor of a quantity is an exact divisor of any number of times this quantity.*

See (69).

126. Lemma.—*A polynomial in its simplest form, the terms of which have no common factor, has no divisor of its own degree, except itself.*

For: Arrange the polynomial according to the descending powers of any letter. If now, the polynomial has a divisor of its own degree (i. e., containing as high a power of the leading letter) with regard to this leading letter, and

the quotient arising from the division *contains* the letter, the product of the divisor and quotient will be *higher in degree* than the dividend, which is absurd. If the quotient *does not contain* the *leading letter* of the divisor, this quotient, multiplied by the several terms of the divisor, will produce the terms of the given polynomial, and will therefore be a *common factor* of all these terms, which is contrary to the original supposition. Since any letter in the given polynomial may be assumed as the *leading letter*, we see that in general a polynomial of the form assumed in the proposition, can have no divisor of its own degree except itself.

127. Now let it be required to find the highest common divisor of P and P' , any polynomials not readily factored. Let the polynomials be arranged according to the descending powers of the same letter. If any factor common to P and P' be perceived, it may be at once taken out and reserved as a factor of the highest common divisor. If any factor be common to the terms of *either* P or P' (but not both) it may be stricken out, as it will have nothing to do with the highest common divisor (121-2). Suppose P' lower in degree than P . If P is exactly divisible by P' , the latter will itself be the highest common divisor of P and P' . For it is not exactly divisible by any polynomial higher in degree, and is its only divisor of its own degree (126). Of course, no polynomial lower in degree than P' could be the highest common divisor, for we would then have a highest common divisor lower than a common divisor. Hence, if P' exactly divides P , it is the highest common divisor of the given polynomials.

Suppose P is not exactly divisible by P' , but contains $P'q$ times, with a remainder of R . Then, by the principles of division, $R = P - P'q$, and hence (124) the highest common divisor of P and P' , which exactly divides P and P' , and therefore $P'q$ (125) exactly divides R . Hence the highest common divisor of P and P' will be the same as the highest common divisor of P' and R . Let all the factors

common to the terms of *either* R or P' (but not *both*), be suppressed, as already explained in speaking of P and P' . If R exactly divides P' , it is the highest common divisor of P' and itself, for, the highest common divisor could not be *lower* in degree than R , a common divisor of itself and P' ; it could not be of a *higher* degree, since it exactly divides R . Hence, it must be of the same degree as R , and therefore (126) must be R itself. Hence, if the remainder R exactly divides the second polynomial, it is the highest common divisor of the given quantities.

If P' is not exactly divisible by R , let a quotient q' and a remainder R'' result from the division. From the principles of division, R'' is equal to $P' - Rq'$, whence the highest common divisor of P' and R (which has already been shown to be the highest common divisor of P and P') must (124) exactly divide R'' . The highest common divisor of P and P' must then be the highest common divisor of P , P' , R and R'' . If now we get the highest common divisor of R'' and R , it will therefore be the highest common divisor of the given polynomials. Suppressing the factors of R and R'' as before, if R'' (which, being a remainder, is lower in degree than R , the divisor producing it) be exactly contained in R , it must, for a reason already given, be the highest common divisor of R and R'' , and therefore of P and P' . In a similar way we may go on and show that any remainder that exactly divides its own divisor, is the highest common divisor of the given polynomials.

Since our reasoning is perfectly general, we deduce, for finding the highest common divisor of two polynomials that cannot be readily factored, the following

RULE.—*Arrange the polynomials, reduced to their simplest forms, according to the powers of the same letter. Reserve all factors appearing in every term of both polynomials, as factors of the highest common divisor. Suppress all factors appearing in every term of either polynomial, but not common to both.*

Divide the polynomial involving the highest power of the leading letter by the other, continuing this division till the term of the remainder involving the highest power of the leading letter is lower in degree in regard to this letter than the corresponding term of the divisor.

Suppress all factors common to all the terms of the remainder, and divide the divisor by the reduced remainder. Treat the resulting remainder and the divisor in the same way, and continue this process until a remainder of zero is obtained. The last divisor, multiplied by the reserved factors, is the highest common divisor sought.

If, at any stage of the process a factor be stricken from both a dividend and a divisor, it must be reserved as a factor of the highest common divisor.

128. Scholium I.—The highest common divisor being the continued product of only the common factors of the given polynomials, it will not be changed if at any time we introduce a factor into one or both of the polynomials, provided we do not introduce the same factor into both the polynomials, or into both the dividend and divisor at any stage of the process. Hence, to avoid fractional terms in the quotients, we may multiply or divide the terms of the dividend or divisor (at any stage of the process) by any quantity, provided we do not multiply both dividend and divisor by the same quantity, and that, if we divide them by the same quantity, we reserve the factors stricken out, for factors of the highest common divisor.

129. Scholium II.—The method (127) will evidently apply to any two polynomials whatever, but it does not save labor if the polynomials can be factored at once.

EXAMPLES.

1. Find the highest common divisor of $x^3 + yx^2 + x^2 - y^2x + 2yx - y^3 + y^2$ and $yx^2 + x^2 + y^2x + yx + x + y$.

We shall find the highest common divisor with respect to x ; hence, arranging according to the descending powers of x , we obtain

$$x^3 + (y+1)x^2 - (y^2 - 2y)x - y^2 + y^2 \quad \big| \quad (y+1)x^2 + (y^2 + y + 1)x + y$$

The first term of the first polynomial not being exactly divisible by $(y+1)x^2$ we multiply it by $y+1$ to render it so divisible, $(y+1)$ not being common to the two polynomials; performing this multiplication and then dividing, in accordance with the rule, we obtain,

$$\begin{array}{r} x^2(y+1) + (y+1)x^2 - (y^2-2y)(y+1)x - y^2(y^2-1) \\ x^2(y+1) + (y^2+y+1)x^2 + xy \hline yx^3 - (y^2-y^2-y)x - y^4+y^2 \end{array} \quad \begin{array}{r} (y+1)x^2 + (y^2+y+1)x + y \\ x+y \end{array}$$

Multiplying this remainder by $(y+1)$ to render its first term exactly divisible by $(y+1)x^2$, we obtain,

$$\begin{array}{r} y(y+1)x^2 - (y^4-2y^2-y)x - y^5-y^4+y^2+y^2 \\ y(y+1)x^2 + (y^2+y^2+y)x \hline (-y^4-y^2+y^2)x - y^5-y^4+y^2 \\ \text{or, } (-y^4-y^2+y^2)x + (-y^4-y^2+y^2)y \end{array}$$

and dividing by the divisor we obtain $+y$ as the second term of the quotient, and $(-y^4-y^2+y^2)(x+y)$ as the remainder. Striking out the factor $-y^4-y^2+y^2$, not found

$$\begin{array}{r} (y+1)x^2 + (y^2+y+1)x + y \\ (y+1)x^2 + (y^2+y)x \hline x+y \\ x+y \hline \end{array} \quad \begin{array}{r} x+y \\ (y+1)x+1 \end{array}$$

in the divisor, there remains $x+y$, and this, being contained exactly in $(y+1)x^2 + (y^2+y+1)x + y$, is the highest common divisor.

Find the highest common divisor of:

2. $x^4-7x^3+8x^2+28x-48$ and $x^3-8x^2+19x-14$.

Ans. $x-2$.

3. $3x^3-13x^2+23x-21$ and $6x^3+x^2-44x+21$.

Ans. $3x-7$.

4. $2x^4-12x^3+19x^2-6x+9$ and $8x^3-36x^2+38x-6$.

Ans. $x-3$.

5. $x^5-2x^4+3x^3-7x^2+8x-3$ and $5x^4-8x^3+9x^2-14x+8$.

Ans. x^2-2x+1 .

6. $x^7+5x^6+6x^5-15x^4+4+8x-6x^4-3x^3$ and $30x^4+30x^4-6x+7x^5-24x^3-45x^2+8$. *Ans.* $x^4+3x^3+x^2-3x-2$.

7. $4xy^2-2y^3+6x^2y$ and $4x^2y+8x^3-4xy^2$. *Ans.* $2(x+y)$.

8. $y^3-5y^2z-4yz^2+2z^3$ and $7y^2z+10yz^2+3z^3$. *Ans.* $y+z$.

9. $x^4-8x^3+21x^2-20x+4$ and $2x^3-12x^2+21x-10$.

Ans. $x-2$.

10. $20x^4y^2-12x^5y^2+(16x^4-15x^3+14x^2)y^2-15xy^2+4y^3$ and $15x^4y^4-9x^3y^4+47x^2y^4-7y^4(3x-4)$. *Ans.* $(5x^2-3x+4)y^2$.

130. III.—To Deduce a Rule for finding the highest common divisor of three or more quantities.

Let P, Q, R, S, represent the given quantities. Let the highest common divisor of P and Q be D. Then D contains all the common factors of P and Q. Let the highest common divisor of D and R be D'. Then D' contains all the common factors of D and R, and therefore of P, Q, and R. If D'' be the highest common divisor of D' and S, it contains all the common factors of D' and S, and therefore of P, Q, R, and S. Similar reasoning might be extended to any number of quantities. Hence the

RULE.—*Find the highest common divisor of two of the quantities, then the highest common divisor of the result and a third, and so on, until all the given quantities have been used. The last highest common divisor will be the highest common divisor of all the given quantities.*

EXAMPLES.

Find the highest common divisor of:

1. $32(x^5-y^5)$, $16x^2-32xy+16y^2$ and $64(x^4-y^4)$

Ans. $16(x-y)$.

2. a^5-b^5 , $a^4-3a^3b+3ab^3-b^4$, and $7a^2-7b^2$. *Ans.* $a-b$.

3. $14ax-8a+ax^2-7ax^2$, $-17a^2x^2+6a^3x^4-28a^2x^2$, and $axm-4am-axn+4an$. *Ans.* $ax-4a$.

SECTION III.

LOWEST COMMON MULTIPLE.

131. A Multiple of a quantity is any quantity exactly divisible by the given quantity.

132. A Common Multiple of two or more quantities is a quantity exactly divisible by every one of the given quantities.

133. The Lowest Common Multiple is the Common Multiple that involves the *lowest exponents* and the *lowest coefficients*.

ILLUSTRATION. mb is a Multiple of b ; 60 of 3, 4, and 5, i. e., a Common Multiple of 3, 4, and 5. [The term *Lowest* is better than *Least*, since an algebraic expression is greater or less, according to the different values assigned to the letters therein.]

134. To Deduce a Rule for finding the Lowest Common Multiple of two or more quantities.

I. Two quantities that may be readily factored.

Take $35a^2b$ and $42ab^4c$. Since the lowest common multiple must be exactly divisible by $35a^2b$, it must contain the factors 7, 5, a , a , b . Since it must contain $42ab^4c$, it must contain the factors 7, 2, 3, a , b , b , b , b , c . Since it is to be the *Lowest* quantity, containing both $35a^2b$ and $42ab^4c$, no factor can enter it that is not absolutely necessary to ensure the divisibility, and no factor oftener than absolutely necessary to accomplish this divisibility. 7, 5, 2, 3, a , b , and c must then enter the lowest common multiple. a must enter twice, or the lowest common multiple will not contain $35a^2b$; b must enter four times or the lowest common multiple will not contain $42ab^4c$. Hence, as these are all the factors necessary to the divisibility of the lowest common multiple by each of the given quantities, the lowest com-

mon multiple is 7. 5. 2. 3. a . a . b . b . b . b . c , or $210a^2b^4c$. As we might treat any two quantities, easily factored, in a similar way, we deduce the following

RULE.—*Separate the given quantities into their simplest factors. Take each factor the greatest number of times it enters any of the quantities. Multiply the results together for the lowest common multiple.*

EXAMPLES.

Find the lowest common multiple of:

1. $84a^2b^3c$ and $108a^3bx^2$.
2. $51a(x+c)^2d$ and $102a^2(x+c)d^2$.
3. $6(a^2-2ab+b^2)$ and $5(a^2+2ab+b^2)$.
4. $3(x^2+y^2)$ and $27a^2b^2(x^2-y^2)$.
5. $17ax^2(x^2+y^2)^2$ and $153a^2x(x^2-y^2)^2$.

135. II.—Two quantities that cannot be readily factored.

Let P and P' be the quantities, and D the highest common divisor. Then, if $\frac{P}{D} = q$ and $\frac{P'}{D} = q'$, $P = qD$ and $P' = q'D$.

The lowest common multiple of P and P' must contain qD , or it will not be divisible by P , and must contain $q'D$ or it will not be exactly divisible by P' . Hence, the fewest factors it can have (since D contains all the factors common to both) are, q , q' and D . Hence, $qq'D = qP'$, is the lowest common multiple.

Since our reasoning is perfectly general, we deduce the following

RULE.—*Find the highest common divisor of the two quantities. Divide the first quantity by this highest common divisor and multiply the quotient by the second quantity. This product is the lowest common multiple.*

136. Scholium.—This case evidently includes (I), in which we can find the highest common divisor by inspection.

137. Corollary.—If the given quantities have no common factors, their highest common divisor is 1. Hence, $\frac{P}{D} = q = P$ and $\frac{P'}{D} = q' = P'$ and $qq'D = P.P.'1$ or $P.P.'$ Hence, if the given quantities have no common factors, the product of the quantities themselves is the lowest common multiple.

EXAMPLES.

Find the lowest common multiple of:

1. $a^2 - 2ax + x^2$ and $a^3 - 3a^2x + 3ax^2 - x^3$.

2. $6x^2 - x - 1$ and $2x^2 + 3x - 2$.

Ans. $(2x^2 + 3x - 2)(3x + 1)$.

3. $y^2 - 12y + 9$ and $y^2 + 9y - 12$.

4. $a^3 + 3a^2b + 3ab^2 + b^3$ and $a^3 + b^3$.

5. $x^4 - 5x^3 + 9x^2 - 7x + 2$ and $x^4 - 6x^2 + 8x - 3$.

Ans. $x^5 - 2x^4 - 6x^3 + 20x^2 - 19x + 6$.

138. III.—Three or more quantities.

Let P, Q, R, S, be the given quantities. Let the lowest common multiple of P and Q be M. Then (135) M must contain all the factors of P and Q. Suppose M' to be the lowest common multiple of M and R; then (135) M' is the lowest quantity that contains all the factors of M and R, and hence of P, Q, and R. Finally, let M'' be the lowest common multiple of M' and S; then it must contain all the factors of M' and S, and hence of P, Q, R, and S; and it must be the lowest quantity, as far as coefficients and exponents are concerned, that does contain all these factors, that is, it must be the lowest common multiple of P, Q, R, and S. The reasoning may evidently be extended to any number of quantities. From the foregoing considerations we deduce the following

RULE.—Find the lowest common multiple of two of the quantities; then the lowest common multiple of this result and a third, and so on, until all the given quantities have been operated upon. The last multiple shall be the lowest common multiple of all the given quantities.

139. Corollary.—If the quantities have no common factor, their product is the lowest common multiple.

EXAMPLES.

Find the lowest common multiple of:

1. d^2-4 , $cd-2c$ and d^2+2d . *Ans.* cd^3-4cd .
2. $7ab(a^2-b^2)$, $3b(a^4-b^4)$ and $42a^3b(a^2-b^2)^2$.
Ans. $42a^3b(a^2+b^2)(a^2-b^2)^2$.
3. $a^2-ab-6b^2$, $a^2+(2b-c)a-2bc$, and $a^2-a(3b+c)+3bc$.
Ans. $a^3-a^2(b+c)-a(6b^2-bc)+6b^2c$.
4. $x^3+xy^2+x^2y+y^3$, $x^3+xy^2-x^2y-y^3$, and x^2-y^2 .
Ans. x^4-y^4 .
5. $a+b$, $a^4+a^2b^2+b^4$, a^6-b^6 , and $a-b$. *Ans.* a^6-b^6 .
6. $x^2+2xc-y^2+c^2$, $x^2+2xy-c^2+y^2$, $x^2-y^2+2yc-c^2$.
7. $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$, $a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$,
and $a^4-a^3b+ab^3-b^4$. *Ans.* a^6-b^6 .
8. $x^4-5x^2+9x^2-7x+2$, $2x^4-12x^2+16x-6$, and $2x^2+2x^2-10x+6$.
Ans. $2x^5-4x^4-12x^3+40x^2-38x+12$.
9. $a^3-6a^2+11a-6$, $a^3-9a^2+26a-24$, and $a^3-8a^2+19a-12$.
Ans. $a^4-10a^3+35a^2-50a+24$.
10. x^3-2x^2+x , x^3-2x+1 , and x^2-1 .

CHAPTER IV.

FRACTIONS.

140. A Fractional Unit is one of the equal parts into which unity, or 1, is divided, as

$$\frac{1}{3}, \frac{1}{w}, \frac{1}{s}.$$

141. A Fraction is a *Fractional Unit*, or a collection of *Fractional Units*, as

$$\frac{1}{4}, \frac{2}{4}, \frac{2}{a}, \frac{b}{5}, \frac{x}{y}.$$

142. The quantity *naming the fractional unit, i. e.*, showing the number of parts into which unity is divided, is called the **Denominator**. The quantity indicating how many fractional units are taken, is called the **Numerator**. The numerator and denominator are together called the **Terms** of the fraction. Thus in $\frac{a}{x}$, a is the *numerator* and x the *denominator*, and a and x are the *terms* of the fraction.

143 A Fractional Expression or Fraction is an indicated division (13), the Numerator, or quantity above the line, corresponding to the dividend, and the Denominator, or quantity below the line, to the divisor. The Quotient arising from the division is called the **Value** of the fraction. Thus

$$\frac{a}{b}, a \div b, a : b, a \overline{)b} \text{ and } \underline{b) a},$$

are equivalent expressions, and the quotient arising from the division is the *value* of the fraction.

144. If the indicated division of (143) can be performed exactly, the quantity expressed is an **Entire** quantity under a *fractional form*. Thus

$$\frac{ab+ac}{a} = b+c.$$

If this indicated division cannot be performed at all, that is, if no terms of the quotient are *entire* or *integral*, the quantity, is a true **Fraction**, as $\frac{x+y}{s}$.

If, again, this indicated division can be partly performed, i. e., if some, but not all, of the terms of the quotient are *entire* or *integral*, the quantity is a **Mixed** quantity under a fractional form, and is usually written as directed in Art. 90. Thus

$$\frac{ax+bx+c}{x} = a + b + \frac{c}{x}.$$

145. Signs.—The sign arising from the combination by (79) of the signs of the numerator and denominator, is the **apparent sign of the fraction**. The sign preceding the vinculum that separates the numerator from the denominator, is the **sign of operation**. The sign resulting from the combination (57) of the **sign of the fraction** and the **sign of operation**, is the **essential or true sign of the fraction**.

146. A fraction is in its **Simplest Form** or **Lowest Terms** when the terms of the fraction have no common factor, or as we sometimes express it, are prime to each other.

For example, $\frac{x}{y}$ is in its simplest form, but $\frac{6x}{12xy}$ is not.

147. The Transformation or Reduction of a fraction is the process of changing its form without altering its value.

148. From (81–83, 143) we have the following propositions:

Proposition I.—*If the Denominator remains unchanged: Multiplying the Numerator by any quantity, multiplies the value of the fraction by the same quantity and dividing the Numerator by any quantity divides the value of the fraction by the same quantity.*

Proposition II.—*If the Numerator remains unchanged: Multiplying the Denominator by any quantity divides the value of the fraction by the same quantity, and dividing the Denominator by any quantity multiplies the value of the fraction by the same quantity.*

Proposition III.—*Multiplying or dividing both numerator and denominator by the same quantity does not alter the value of the fraction.*

149. From these propositions we derive the following corollaries:

Corollary I.—*Changing the sign of either the numerator or the denominator changes the sign of the fraction; but changing the sign of both numerator and denominator does not alter the sign of the fraction.*

Changing the sign of a term is the same as multiplying or dividing it by -1 .

Corollary II.—*A fraction may be multiplied by multiplying the numerator or by dividing the denominator, and a fraction may be divided by dividing the numerator or by multiplying the denominator.*

Scholium.—Whatever term is operated upon, the other is supposed to remain unchanged.

Corollary III.—*A factor may be introduced into, or rejected from, both terms of a fraction, without altering the value of the fraction.*

Scholium.—The process of rejecting, or dividing out, a factor common to the two terms of a fraction is called **Cancellation**.

General Scholium.—A CHANGE IN THE NUMERATOR PRODUCES A LIKE CHANGE IN THE VALUE OF THE FRACTION; A CHANGE IN THE DENOMINATOR PRODUCES A CHANGE OF THE OPPOSITE CHARACTER IN THE VALUE OF THE FRACTION.

(A.) TRANSFORMATIONS.

I. To Deduce a Rule for reducing a fraction to its Simplest Form.

150. Since a fraction is in its simplest form when the numerator and denominator have no common factors, and since (149, III.) we may reject, or divide out, any factor common to the two terms, we may reduce a fraction to its simplest form by dividing out or cancelling all the factors common to the two terms. As the highest common divisor of the terms is the product of all the factors common to them, we may operate more directly by simply dividing the terms by their highest common divisor. Hence, the

RULE.—*Cancel all factors common to the numerator and denominator, or divide both terms of the fraction by their highest common divisor.*

EXAMPLES.

Reduce the following fractions to their lowest terms:

$$1. \frac{96a^8b^6c^3d}{128a^{10}b^{12}c^5d^2f}, \quad \frac{5cd(x+y)^2}{25c^2d^2(x+y)^4}$$

$$2. \frac{(x-y)^2}{x^2-3x^2y+3xy^2-y^2}$$

$$Ans. \frac{1}{x-y}$$

$$3. \frac{x-y}{x^2-y^2}$$

$$Ans. \frac{1}{x+xy+y^2}$$

$$4. \frac{3a+4bc}{27a^3+64b^3c^3}$$

$$Ans. \frac{1}{9a^2-12abc+16b^2c^2}$$

$$5. \frac{ac+bd+ad+bc}{af+2bx+2ax+bf}$$

$$Ans. \frac{c+d}{f+2x}$$

$$6. \frac{y^2+(3a+2b)y+6ab}{y^2+(4c+2b)y+8bc}$$

$$Ans. \frac{y+3a}{y+4c}$$

$$7. \frac{3bd-2d-3bc+2c}{3b^2-3bd-2b+2d} \quad \text{Ans. } \frac{c-d}{d-b}.$$

$$8. \frac{6a^4b-6a^2b+12a^2b^3}{8a^3b^3-8ab^3+16ab^4} \quad \text{Ans. } \frac{3a}{4b}.$$

$$9. \frac{7ab^3+7a}{20b^3-20b+20} \quad \text{Ans. } \frac{7a(b+1)}{20}.$$

$$10. \frac{3x-x^2+y(x-2)-2}{(4+x)y-5(x-1)-y^2} \quad \text{Ans. } \frac{x-2}{5-y}.$$

$$11. \frac{14xy-7xw+14yz-7ws}{14xy+14yz+7w(x+s)} \quad \text{Ans. } \frac{2y-w}{2y+w}.$$

151. II.—To Deduce a Rule for Reducing a quantity from a Fractional form to the form of an Entire or Mixed quantity.

From (144) we see that this transformation is the mere performance, so far as possible, of an indicated operation. Hence the

RULE.—*Divide the Numerator by the Denominator. If the division is not EXACT, place the remainder over the divisor, at the right of the quotient, with a vinculum separating the remainder from the divisor.*

EXAMPLES.

$$1. \frac{x^2+y^2}{x-y} \quad \text{Ans. } x^2+xy+y^2+\frac{2y^2}{x-y}.$$

$$2. \frac{\frac{1}{2}x^3+x^2+\frac{3}{8}x+\frac{3}{4}+m}{\frac{1}{2}x+1} \quad \text{Ans. } x^2+\frac{3}{4}+\frac{m}{\frac{1}{2}x+1}.$$

$$3. \frac{a^2+2c-b^2}{a-b} \quad \text{Ans. } a+b+\frac{2c}{a-b}.$$

$$4. \frac{6x^2+22xy+21c+20y^2}{2x+4y} \quad \text{Ans. } 3x+5y+\frac{21c}{2(x+2y)}.$$

$$5. \frac{x^2 - y^2}{x^4 + x^2 y^2 + y^4}. \quad \text{Ans. } x^2 - y^2.$$

$$6. \frac{6x - 3xy - 3y + 6}{3(1+x) + xy + y}. \quad \text{Ans. } 2 - \frac{5y}{3+y}.$$

152. Conversely.—To reduce an Entire or Mixed quantity to the form of a fraction.

Here we have given (143-44) the value of the fraction, corresponding to the quotient, i. e., the given Entire or Mixed quantity, and the divisor is the denominator of the fraction required. Since the dividend equals the product of the quotient and divisor, increased by the remainder, and since the numerator of the fraction required corresponds to the dividend, we have the following

RULE.—Multiply the entire or integral part of the given quantity by the given denominator. To this product add the numerator of the fractional part, and place the result over the given denominator.

153. Scholium.—If the quantity to be reduced is Entire, the denominator of the required fraction is *given explicitly*. If the quantity to be reduced is Mixed, the denominator of the fractional part is what is called the *given denominator*.

EXAMPLES.

$$1. 7a + 2b + \frac{3c}{a+b}. \quad \text{Ans. } \frac{7a^2 + 9ab + 2b^2 + 3c}{a+b}$$

$$2. 8x + 7y + \frac{3-x}{2x+y}.$$

$$3. x^2 - x^2 y + xy^2 - y^2 + \frac{3x}{x+y}. \quad \text{Ans. } \frac{x^4 - y^4 + 3x}{x+y}.$$

$$4. 9a - 12b - c + \frac{4a-c}{3a+2b}.$$

$$5. a + b - c + \frac{d}{a+b+c}. \quad \text{Ans. } \frac{a^2 + 2ab + b^2 - c^2 + d}{a+b+c}.$$

$$6. 4a^2 - 16ax + 8x^2 + \frac{3a^2x^2 - d}{5a^2 - 2a^2x}.$$

$$\text{Ans. } \frac{20a^3 - 88a^2x + 50a^2x^2 - 6a^2x^3 - d}{5a^2 - 2a^2x}.$$

154. III.—To Deduce a Rule for reducing a fraction having any denominator, to an equivalent fraction having a given denominator.

Let it be required to reduce $\frac{a}{b}$ to an equivalent fraction having the denominator c . Now, to reduce b to c , we may multiply it by c , giving us bc , and then divide the result by b , giving us $\frac{bc}{b}$ or c . Hence, we must multiply and divide a by the same quantities, that the value of the fraction may not be altered. Hence, we must multiply a by c , giving us ac , and divide this result by b , giving us $\frac{ac}{b}$, for a new numerator. Then the new fraction has $\frac{ac}{b}$ for its numerator and c for its denominator. Now, if we suppose $\frac{a}{b}$ to be in its simplest form, ac cannot be exactly divisible by b , unless c is so divisible; for a and b have no common factors. That is, in order that both terms of the new fraction may be entire, the new denominator, c , must be some *multiple* of b . If, then, c is a multiple of b , let $\frac{c}{b} = m$.

Then $\frac{ac}{b} = am =$ the numerator of the new fraction.

This fraction will be, then, $\frac{am}{c}$.

As an illustration, if we are required to reduce $\frac{2}{3}$ to an equivalent fraction having the denominator 27, we have $\frac{27}{3} = 9$, which is the m of our general discussion. Hence

the new fraction, $\frac{am}{c} = \frac{2 \times 9}{27} = \frac{18}{27}.$

From the foregoing considerations, we deduce the following

RULE.—*Divide the denominator of the required fraction by the denominator of the given fraction. Multiply the numerator of the given fraction by the quotient, and place the result over the new denominator.*

EXAMPLES.

1. Reduce $\frac{a+b}{a-b}$ to a fraction with the denominator $(a-b)^2$.

$$\text{Ans. } \frac{a^2-b^2}{(a-b)^2}.$$

2. Transform $\frac{2a+3b}{3a-b}$ that the new denominator may be $3a^2-7ab+2b^2$.

$$\text{Ans. } \frac{2a^2-ab-6b^2}{3a^2-7ab+2b^2}.$$

3. Transform $\frac{x-2y}{x+y-z}$ that the new denominator may be $x^2+y^2+2xy-z^2$.

4. Transform $\frac{x+5a}{x+7b}$ that the new denominator may be $x^2+(7b+8c)x+56bc$.

155. IV.—**To Deduce a Rule for reducing fractions having different denominators to equivalent fractions having a common denominator.**

Let any quantity be assumed for a common denominator. Then by (154) we may reduce each of the given fractions to an equivalent fraction having this denominator. In order that the new numerators may all be entire or integral, the denominator assumed must, as explained (154) be a multiple of each of the given denominators, i. e., a common multiple of them. And in order that the new numerators may be as simple as possible we assume the lowest common multiple for the common denominator. From these considerations we deduce, for reducing frac-

tions having different denominators to equivalent fractions having a common denominator, the following

RULE.—Assume the lowest common multiple of the given denominators for the common denominator. Divide this lowest common multiple by each denominator separately, and multiply the result by the corresponding numerator. The resulting products will be the new numerators.

156. Scholium.—When we assume simply a common multiple for the common denominator we are often said to reduce the fractions to equivalent ones having a *common denominator*. When we assume the lowest common multiple we are said to reduce them to equivalent fractions having the *lowest or least common denominator*.

EXAMPLES.

Reduce, in each example, the fractions to equivalent ones having the lowest common denominator.

$$1. \frac{5a}{3b}, \frac{4c}{6b^2c}, \frac{9ab}{17cf}, \text{ and } \frac{3x}{4b^2c}.$$

$$2. \frac{1}{x^4+x^2y+x^2y^2+xy^3+y^4}, \frac{3a}{x-y}, \text{ and } \frac{4c}{x^2-y^2}.$$

$$\text{Ans. } \frac{x^2-y^2}{x^6-xy^5+x^5y-y^6}, \frac{3a(x^4+x^2y+x^2y^2+xy^3+y^4)(x+y)}{x^6-xy^5+x^5y-y^6},$$

$$\text{and } \frac{4c(x^4+x^2y+x^2y^2+xy^3+y^4)}{x^6-xy^5+x^5y-y^6}.$$

$$3. \frac{a}{x-y}, \frac{b}{x+y}, \frac{c}{x^2+xy+y^2}, \text{ and } \frac{d}{x^2-xy+y^2}.$$

$$4. \frac{a}{a-b}, \frac{a^2}{a^2-b^2}, \text{ and } \frac{a^4}{a^4-b^4}.$$

$$\text{Ans. } \frac{a^4+a^2b+a^2b^2+ab^3}{a^4-b^4}, \frac{a^4+a^2b^2}{a^4-b^4}, \text{ and } \frac{a^4}{a^4-b^4}.$$

$$5. \frac{a}{x-1}, \frac{b}{x^2+1}, \frac{c}{x^2-1}, \frac{d}{x^4-1}, \text{ and } \frac{e}{x^6-1}.$$

157. V.—To Deduce a Rule for transferring a factor from either term of a fraction to the other.

(Given the fraction $\frac{a^m b^{-n}}{c^{-r} d^s}$. Let it be required to transfer a^m to the denominator and c^{-r} to the numerator. Multiplying both terms of the given fraction by $a^{-m} c^r$ (recollecting that $a^{-m} \times a^m = \frac{1}{a^m} \times a^m = 1$, and $c^{-r} \times c^r = \frac{1}{c^r} \times c^r = 1$, (149)), we obtain $\frac{1 \times b^{-n} c^r}{1 \times d^s a^{-m}}$ or $\frac{b^{-n} c^r}{d^s a^{-m}}$. Here we see that the required transfer has been made, and that the sign of the exponent of each factor transferred has been changed. As these letters may represent any quantities whatever, we conclude that in general we may transfer a factor from either term of a fraction to the other by simply making the change and giving to the exponent of every factor transferred a sign opposite to the sign of the exponent of the same factor in the original fraction. Hence the

RULE.—Place the factors to be transferred in the terms to which they are to be transferred, changing the sign of the exponent of every factor transferred.

EXAMPLES.

1. Transfer all the factors of each term of the fraction to the other:

$$\frac{a^5 b^6 c^3}{d^2 e^2 f^2}, \quad \frac{a m b (x+y)^6}{c r d (x+y)^6}, \quad \frac{a^2 + 7xy}{(a^2 - 7xy)y}, \quad \text{and} \quad \frac{b+cy}{(a+c)x}.$$

2. Transfer x , y , z , and w , to the numerator, and a , b , and c , to the denominator, in each of the following fractions:

$$\begin{aligned} & \frac{a^2 b^3 (a+c)^5 x}{d^2 c^2 (y+x)^m} & \text{Ans. } \frac{x(y+x)^{-m}}{d a^{-2} b^{-3} c^2 (a+c)^{-5}} \\ & \frac{x(b+a+c)^n (y+x)}{a(x+y+z)^6 (b+c)} & \text{Ans. } \frac{x(y+x) (x+y+z)^{-n}}{a(b+c) (b+a+c)^{-6}} \\ & \frac{ab(c+a)^m (b+c)^2 x}{(a^2+b^2)^2 (x+y+z+w)^3} \end{aligned}$$

(B.) OPERATIONS ON FRACTIONS.**158. I.—Addition.****To Deduce a Rule for Adding Fractions.**

Since only things of the same kind can be added, and since the denominator of the fractional unit names the units, *i. e.*, tells the kind of things taken, we can add fractions only when the fractional units are the same; in other words, only when the denominators are alike. As the numerator of each fraction shows how many times the fractional unit is taken in that fraction, the sum of all the numerators shows how many times the fractional unit is taken in all the fractions. Hence this sum, placed as a numerator over the common denominator, is the simplest expression for the aggregate of the given fractions: is their sum.

Now, if the given fractions have not the same fractional unit, we may (155) transform them so that the fractional units in the resulting fractions will be identical. The sum of the given fractions will then be the same as the sum of their equivalents, which may be found as just indicated.

From the foregoing considerations we deduce, for adding fractions, the following

RULE.—*Reduce the fractions to equivalent fractions having a common denominator. Add the numerators and place the result as a numerator over the common denominator.*

159. Scholium.—Since the whole equals the sum of its parts, we may add mixed quantities by taking the sum of the entire or integral, and of the fractional parts, separately, and then uniting these by the proper sign; or we may reduce given quantities to equivalent quantities expressed in fractional form, and then apply the rule of (158).

EXAMPLES.

1. $\frac{1}{a} + \frac{a}{a+b} + \frac{b}{b+c} = ?$ (2.) $\frac{a}{x} + \frac{b-c}{y} + \frac{x}{a-b} = ?$
3. $\frac{8a-11b}{12(ab-b^2)} + \frac{2a+b}{3a^2-3ab} + \frac{3}{4b-4a} = ?$ *Ans.* $\frac{2a-b}{3ab}.$
4. $\frac{3x^2+y^2}{4(x^2-y^2)} + \frac{2x+y}{6(y-x)} + \frac{2y-x}{6(y+x)} = ?$ *Ans.* $\frac{1}{4}.$
5. $\frac{2b}{(a-b)(a+b)} + \frac{1}{a+b} + \frac{c}{a-b} = ?$ *Ans.* $\frac{1+c}{a-b}.$
6. $\frac{a^2-x^2}{ax} + \frac{x-a}{x} + \frac{x}{a} = ?$ *Ans.* 1.
7. $\frac{3x+y}{3x^2-xy} + \frac{2x+3y}{4(y^2-3xy)} + \frac{3x^2-xy-6y^2}{18x^2y-6xy^2} = ?$ *Ans.* $\frac{25}{12(3x-y)}.$
8. $\frac{1}{x-1} + \frac{2}{x^2-1} + \frac{4}{x^4-1} = ?$ *Ans.* $\frac{x^3+3x^2+x+7}{x^4-1}.$
9. $\frac{3x}{4x-4y} + \frac{x^2+5xy}{4y^2-4x^2} + \frac{2x^2+xy}{2xy+y^2} = ?$ *Ans.* $\frac{2x^2+3xy}{2xy+2y^2}.$
10. $\frac{(x+1)^2}{x^2-1} + \frac{x^2-1}{(x+1)^2} = ?$ *Ans.* $2\frac{x^2+1}{x^2-1}.$
11. $\frac{3a^4}{2b(3a^2+b^2)} + \frac{(4a^2+b^2)b}{2(3a^2+b^2)} = ?$ *Ans.* $\frac{a^2+b^2}{2b}.$
12. $\frac{x}{3} + \frac{x+1}{4} + \frac{x-1}{x+1} = ?$ *Ans.* $\frac{7x^2+22x-9}{12x+12}.$
13. $\frac{1}{(a+x)^2} + \frac{1}{a^2-x^2} + \frac{5y}{a^4-x^4} = ?$ *Ans.* $\frac{2a(a^2+x^2)+5y(a+x)}{a^6-ax^4+a^4x-x^6}.$

$$14. \frac{a^2+x^2}{a^2-x^2} + \frac{a^2-x^2}{a^2+x^2} = \frac{(a^2+x^2)^2 + (a^2-x^2)^2}{a^4-x^4} = \frac{2(a^4+x^4)}{a^4-x^4}.$$

$$15. \frac{a}{a+x} + \frac{a}{a-x} = \frac{a^2-ax}{a^2-x^2} + \frac{a^2+ax}{a^2-x^2} = \frac{a^2-ax+a^2+ax}{a^2-x^2} = \frac{2a^2}{a^2-x^2}$$

II. Subtraction.

160. To Deduce a Rule for subtracting one fraction from another.

Since things can be subtracted only from things of the same kind as themselves, we transform the given fractions (if necessary) so that they may have the same unit. Then since, by definition, the difference or remainder is the quantity which, added to the subtrahend, produces the minuend, the numerator of the fraction indicating the difference between the given fractions must (158), when added to the numerator of the subtrahend, equal the numerator of the minuend. Hence, to get the numerator of the difference we take the difference between the numerators of the given fractions. From the foregoing considerations we deduce, for subtracting one fraction from another, the following

RULE.—*Transform the fractions so that the denominators may be equal. Subtract the numerator of the subtrahend from the numerator of the minuend. Place this difference, as a numerator, over the common denominator.*

161. Scholium I.—From (159) it follows that to subtract one mixed quantity from another, we may subtract the entire or integral part of the subtrahend from the corresponding part of the minuend, and then subtract the fractional part of the subtrahend from the corresponding part of the minuend and unite the results. Or, we may reduce the given quantities to their equivalents expressed fractionally, and then apply the rule (160).

162. Scholium II.—In adding and subtracting fractions we should remember that the line separating the numerator from the denominator is a vinculum, and therefore a sign prefixed to this vinculum must be considered as applying to *every term* of the numerator, or of the denominator. If applied to the terms of both, of course the effect of the sign would be destroyed (149).

EXAMPLES.

1. From $\frac{a+x}{a-x}$ take $\frac{a-x}{a+x}$.

$$\frac{a+x}{a-x} - \frac{a-x}{a+x} = \frac{(a+x)^2}{a^2-x^2} - \frac{(a-x)^2}{a^2-x^2} = \frac{a^2+2ax+x^2-a^2+2ax-x^2}{a^2-x^2} = \frac{4ax}{a^2-x^2}.$$

$$2. \frac{a^2+x^2}{a^2-x^2} - \frac{a^2-x^2}{a^2+x^2} = \frac{(a^2+x^2)^2}{a^4-x^4} - \frac{(a^2-x^2)^2}{a^4-x^4} = \frac{a^4+2a^2x^2+x^4-a^4+2a^2x^2-x^4}{a^4-x^4} = \frac{4a^2x^2}{a^4-x^4}.$$

3. From $2x + \frac{5x-2}{7}$ take $3x - \frac{4x+5}{6}$. Ans. $\frac{16x+23}{42}$.

4. From $\frac{x+y}{12}$ take $\frac{x-y}{12}$. Ans. $\frac{y}{6}$.

5. $\frac{x}{2x-y} - \frac{4yx}{4x^2+4xy-3y^2} = ?$ Ans. $\frac{x}{2x+3y}$.

6. $10x + \frac{14x-13}{20} - \left(4x + \frac{2x-5}{4}\right) = ?$ Ans. $6x + \frac{x+3}{5}$.

7. $\frac{y}{y-7} - \frac{y+7}{y} = ?$

$$8. \frac{x^2+y^2}{x^2-y^2} - \frac{y}{x-y} = \frac{x^2+y^2}{x^2-y^2} - \frac{y(x+y)}{x^2-y^2} = \frac{x^2+y^2-(yx+y^2)}{x^2-y^2} = \frac{x^2-yx}{x^2-y^2} = \frac{(x-y)x}{(x-y)(x+y)} = \frac{x}{x+y}.$$

$$9. \frac{3x}{4} - \left(\frac{5x}{8} - \frac{2x}{5} \right) = ?$$

$$\text{Ans. } \frac{21x}{40}.$$

$$10. \frac{x+y^2}{xy-y^2-xz+yz} - \frac{x+xy}{x^2-xy+yz-xz} - \frac{yx-x}{x^2-(y+x)z+xy} = ?$$

$$11. \frac{9a^2-11b^2}{12a^2-6b^2} - \left(\frac{a^2-b^2}{2(2a^2+b^2)} + \frac{a^2b^2-7b^4}{12a^4-3b^4} \right) = ? \quad \text{Ans. } \frac{a^2}{2a^2+b^2}.$$

$$12. \frac{a}{2b} + \frac{a^2+6ab}{6(ab+b^2)} - \left(\frac{4a^2-ab}{6(ab-b^2)} - \frac{2a^2+ab}{3(a^2-b^2)} \right) = ? \quad \text{Ans. } \frac{a}{a+b}.$$

13.

$$\frac{x}{x^2-y^2} + \frac{y}{x+y} + \frac{1}{x-y} - \left(\frac{(3x^2+y^2)2}{3(x^2+y^2)} - \frac{6x^2-y^2}{6(x^2-y^2)} + \frac{19x^2y^2+3y^4}{6(x^4-y^4)} \right) = ?$$

$$\text{Ans. } \frac{2x+y+xy-(x^2+y^2)}{x^2-y^2}.$$

163. III.—Multiplication.

To Deduce a Rule for Multiplying one fraction by another.

Let it be required to multiply $\frac{a}{b}$ by $\frac{c}{d}$. This means that we are to multiply the fraction $\frac{a}{b}$ by c after the division of c by d . If we multiply $\frac{a}{b}$ by c (149) we obtain $\frac{ac}{b}$. We were required, however, to multiply $\frac{a}{b}$, not by c , but by $\frac{c}{d}$. Hence we have multiplied by a quantity d times too large. The result is therefore d times too large, and we find the desired product by dividing this result by d , thus obtaining (149) $\frac{ac}{bd}$. A review of the work shows that this product may be obtained at once by multiplying together the numerators of the given fractions for a new numerator, and the denominators for a new denominator. As the letters employed may represent any quantities whatever, and the reasoning may

obviously be extended to the multiplication of any number of fractions, we deduce, for multiplying fractions, the following

RULE.—*Multiply the numerators together for the numerator of the product. Multiply the denominators together for the denominator of the product.*

164. Corollary.—*To multiply any quantity whatever by a fraction, multiply that quantity by the numerator of the fraction, and divide the result by the denominator of the multiplier.*

EXAMPLES.

1. Multiply $\frac{a-b}{c}$ by $\frac{a+b}{d}$; $\frac{a+7x}{b}$ by $\frac{a-7y}{c}$.

2. Multiply $\frac{b}{a+b}$, $\frac{a^2-b^2}{b^3}$, $\frac{a}{a-b}$, and $\frac{b}{a}$ together.

Ans. 1.

3. Multiply $\frac{b^2+1}{x+a}$, $\frac{x-a}{b^2-b+1}$, and $\frac{x+a}{b+1}$ together.

4. Multiply $\frac{5x}{2x+3y}$ by $\frac{3x-4y}{12xy}$. Ans. $\frac{15x-20y}{24xy+36y^2}$.

5. Multiply $\frac{7ax}{b}$, $\frac{3a(x+y)}{x-y}$, and $\frac{6b(x-y)}{x^2-y^2}$ together.

Ans. $\frac{126a^2x}{x-y}$.

6. Multiply $\frac{a^2+2ab+b^2-c^2}{a+b+c}$ by $\frac{a^2+2ab+b^2+c^2}{a+b-c}$.

Ans. $a^2+2ab+b^2+c^2$.

7. Multiply $\frac{7+3x}{40}$, $\frac{3c(x+y)}{7}$, and $\frac{49(x^2-y^2)}{14+6x}$ together.

8. Multiply $\frac{a+x}{a-x} + \frac{a-x}{a+x}$ by $\frac{a+x}{a-x} - \frac{a-x}{a+x}$.

This product equals $\frac{2(a^2+x^2)}{a^2-x^2} \cdot \frac{4ax}{a^2-x^2} = \frac{8ax(a^2+x^2)}{(a^2-x^2)^2}$.

9. Multiply $\frac{x^2+bx}{x-b}$ by $\frac{x^2-b^2}{x+b}$. *Ans.* $x(x^2+bx+b^2)$.

10. Multiply $x+\frac{x+1}{a}$ by $\frac{x-1}{a+b}$. *Ans.* $\frac{(ax+x+1)(x-1)}{a(a+b)}$.

11. Multiply $\frac{x^2+10x+16}{x^2+16x+28}$ by $\frac{x^2+19x+70}{x^2+20x+96}$. *Ans.* $\frac{x+5}{x+12}$.

12. Multiply $\frac{a}{a+b}+\frac{b}{a-b}$ by $\frac{a}{a-b}\frac{b}{a+b}$.
Ans. $1+\frac{4a^2b^2}{(a^2-b^2)^2}$.

165. IV.—Division.

To Deduce a Rule for dividing a fraction by a fraction.

Let it be required to divide $\frac{x}{y}$ by $\frac{m}{n}$. If we divide $\frac{x}{y}$ by m , we obtain (149) $\frac{x}{my}$. We were not, however, required to divide $\frac{x}{y}$ by m , but by m after it had been divided by n .

Hence, as we have divided by a quantity n times too large, the quotient must be n times too small, and to obtain the true result from dividing $\frac{x}{y}$ by $\frac{m}{n}$, we must multiply the first quotient, $\frac{x}{my}$, by n .

Doing so, we obtain (149) $\frac{nx}{my}$. A comparison of the result and the given fractions shows that the quotient might have been obtained at once by inverting the divisor and proceeding as in multiplication (163). As the letters employed may represent any quantities whatever, we deduce, for dividing a fraction by a fraction, the following

RULE.—*Invert the divisor and proceed as in multiplication.*

166. Cor. I.—*The reciprocal of a fraction is the fraction inverted.*

167. Cor. II.—*Multiplying by a quantity is the same as dividing by its reciprocal, and dividing by a quantity the same as multiplying by its reciprocal.*

168. Scholium.—The indicated division of one fraction

by another $\left(\frac{\frac{a}{b}}{\frac{x}{y}}\right)$ is often called a **Complex fraction**. The

process of performing the division is then called *simplifying the fraction*, i. e., reducing it to a fraction, both terms of which are integral. Fractions, both terms of which are integral, are called *simple fractions*.

EXAMPLES.

1. Divide $\frac{ab+bx}{x}$ by $\frac{ab}{x}$. Ans. $1 + \frac{x}{a}$.

2. Divide $\frac{x^4-b^4}{b^3c+bc^3}$ by $\frac{x^2+b^2}{b+c}$. Ans. $\frac{x^2-b^2}{bc}$.

3. Divide $\frac{x+2}{x+3}$ by $\frac{x^2+5x+4}{x^2+7x+12}$. Ans. $\frac{x+2}{x+1}$.

4. Divide $\frac{x^4-b^4}{x^3-2bx+b^3}$ by $\frac{x^2+bx}{x-b}$.

This quotient equals

$$\frac{x^4-b^4}{(x-b)^2} \cdot \frac{x-b}{(x+b)x} = \frac{(x^2-b^2)(x^2+b^2)(x-b)}{(x-b)(x-b)(x+b)x} = \frac{x^2+b^2}{x}.$$

5. Divide

$$\frac{2-3x+x^2}{1-2x-x^2} + \frac{x+3x^2}{x^2+2x-1} \text{ by } \frac{\frac{-64x^2+28xy+5y^2}{(4x+y)^2} + \frac{16x-5y}{4x+y}}{\frac{2x-4y}{3x-3y} - \frac{y^2-5xy}{6x^2-6xy} - \frac{2x}{16x^2-y^2}}$$

Performing the operations indicated in each term of the first fraction, that fraction becomes

$$\frac{2}{4x+y} - \frac{12x}{4x+y} = \frac{-10x}{4x+y}$$

Reducing the second in a similar way, it becomes

$$\frac{\frac{24xy}{(4x+y)^2}}{\frac{2x}{16x^2-y^2}} = \frac{24xy}{(4x+y)(4x+y)} \cdot \frac{(4x+y)(4x-y)}{2x} = \frac{12y(4x-y)}{4x+y}$$

$$\frac{12x}{4x+y} \div \frac{12y(4x-y)}{4x+y} = \frac{x}{y(4x-y)} \quad \text{Ans.}$$

6. Divide $1 + \frac{n-1}{n+1}$ by $1 - \frac{n-1}{n+1}$. Ans. n .

7. Divide $\frac{1+x^2}{1-x^2}$ by $\frac{1-x^2}{1+x^2}$. Ans. $\frac{(1+x^2)^2}{(1-x^2)^2}$.

8. Divide $\frac{a+x}{a-x} + \frac{a-x}{a+x}$ by $\frac{a^2+x^2}{2ax}$. Ans. $\frac{4ax}{a^2-x^2}$.

9. Divide $\frac{a+b}{x+y}$ by $\frac{(a+b)^5}{(x+y)^5}$. Ans. $\frac{(x+y)^6}{(a+b)^4}$.

10. Divide $\frac{1-\frac{3}{a}}{1-\frac{1}{x}+\frac{1}{x^2}}$ by $\frac{a^2-6a+9}{x^2-x+1}$. Ans. $\frac{x^2}{a(a-3)}$.

11. Divide $\frac{x^2}{x^2-y^2}-1$ by $\frac{y^2}{x^2-y^2}+1$. Ans. $\frac{y^2}{x^2}$.

12. Divide $\frac{1}{x} + \frac{1}{xy^2}$ by $y + \frac{1}{y} - 1$. Ans. $\frac{y+1}{xy^2}$.

(C.) CONTINUED FRACTIONS.

169. A Continued Fraction is a fraction that has an integer for a numerator, and for denominator an integer plus a fraction that has an integral numerator and for denominator an integer plus a fraction, and so on.

170. The continued fractions that occur most frequently, and the only ones that will be treated here, are those having 1 for a numerator and for denominator an integer plus a

fraction that has 1 for its numerator and for denominator an integer plus a fraction, and so on.

ILLUSTRATION. $\frac{1}{2 + \frac{1}{3 + \frac{1}{4 + \frac{1}{5}}}}$, $\frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \frac{1}{e}}}}}$,

are continued fractions.

171. The separate fractions $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, etc., that make up the continued fraction, are called **Integral fractions**.

172. A **Terminating continued fraction** is one in which the number of integral fractions is finite; it is a fraction that terminates in finite numbers, as $\frac{1}{2 + \frac{1}{3 + \frac{1}{4}}}$.

173. An **Infinite continued fraction** is one in which the number of integral fractions is infinite; it is one that does not terminate in finite quantities.

174. A **Periodic continued fraction** is one in which the denominators, often after a certain number of changes, are repeated in the same order.

The fractions may be periodic from the beginning, or after a certain number of changes, and the denominators may recur periodically in pairs, sets of three, four, etc., to n .

ILLUSTRATION.

$$\frac{1}{a + \frac{1}{a + \frac{1}{a + \frac{1}{a}}}}, \frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{c + \frac{1}{b + \frac{1}{a + \frac{1}{b + \frac{1}{c}}}}}}}}, \frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{b + \frac{1}{c + \frac{1}{b + \frac{1}{c}}}}}}}, \frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{b + \frac{1}{c}}}}}}}}$$

175. To transform a common fraction into a continued fraction.

Take the common fraction $\frac{4}{7}$, which is in its simplest form. Since the terms may be divided by the same quantity without altering the value of the fraction, we may divide both terms by 4, thus obtaining $\frac{1}{1+\frac{3}{4}}$; again, dividing both terms of $\frac{3}{4}$ by 3 we obtain $\frac{3}{4} = \frac{1}{1+\frac{1}{3}}$.

$$\text{Hence, } \frac{4}{7} = \frac{1}{1 + \frac{1}{1 + \frac{1}{3}}}$$

In general, let $\frac{a}{b}$ represent any common fraction in its simplest form. Since both terms may be divided by the same quantity without altering the fraction, both terms may be divided by a without altering the fraction. Suppose a is contained in b , q times, with a remainder r ; then $\frac{a}{b} = \frac{1}{q + \frac{r}{a}}$; again, dividing a by r , suppose the quotient q'

with the remainder r' results, so that

$$\frac{r}{a} = \frac{1}{q' + \frac{r'}{r}}; \text{ then } \frac{a}{b} = \frac{1}{q + \frac{1}{q' + \frac{r'}{r}}}$$

Operating on $\frac{r'}{r}$ as on the preceding fractions, obtaining a quotient q'' and remainder r'' ,

$$\frac{r'}{r} = \frac{1}{q'' + \frac{r''}{r'}} \quad \text{and} \quad \frac{a}{b} = \frac{1}{q + \frac{1}{q' + \frac{1}{q'' + \frac{r''}{r'}}}}$$

and this process may evidently be continued as long as we please. The operations performed may be summarized thus:

$$\begin{array}{r} a) \ b \ | \ q \\ \underline{aq} \\ r) \ a \ | \ q \\ \underline{rq'} \\ r') \ r \ | \ q'' \\ \underline{r'q''} \\ r'' \text{ etc.,} \end{array}$$

they are evidently those required in finding the highest common divisor of a and b ; and as the reasoning is perfectly general, we have the following

RULE.—*Apply to the terms of the given fraction, in its simplest form, the rule for finding their highest common divisor. The successive quotients will be the denominators of the successive integral fractions, the numerators being always unity.*

As the terms of the given fraction have no common divisor except unity, the application of the foregoing rule must lead to a remainder of 1 at some stage of the process, at which point the continued fraction terminates. *A common rational fraction can therefore always be reduced to a TERMINATING continued fraction.*

176.—To transform a continued fraction into a common fraction.

I.—The continued fraction is finite, or terminating.

It is clear that the performance of the operations indicated in the continued fraction will not alter the value of the fraction. Hence, by performing these operations, *i. e.*, reversing the process indicated in (175), we shall arrive at a common fraction equivalent to the continued fraction.

II.—The fraction is infinite, *i. e.*, does not terminate.

Only when all the integral fractions are considered can a fraction be obtained *exactly* equivalent to the continued fraction. If the number of integral fractions is infinite, as we

cannot operate upon them all, we can obtain only approximate results, and these will be closer approximations the greater the number of integral fractions considered.

An Approximating Fraction or Convergent is the result obtained by stopping at any integral fraction and neglecting all that follow.

ILLUSTRATION. $\frac{1}{a}, \frac{1}{a + \frac{1}{b}} = \frac{b}{ab+1}, \frac{1}{a + \frac{1}{b + \frac{1}{c}}} = \frac{bc+1}{(ab+1)c+a}$

etc., are approximating fractions.

The Order of a Convergent is indicated by the number of integral fractions taken to obtain the convergent.

ILLUSTRATION.—The convergents $\frac{1}{a}, \frac{1}{a + \frac{1}{b}}, \frac{1}{a + \frac{1}{b + \frac{1}{c}}}$ are of the first, second, and third order, respectively.

177. Theorem.—*The true value of the continued fraction is always intermediate between the values of any two successive approximating fractions.*

Denote the true value of the continued fraction

$$\frac{1}{a + \frac{1}{b + \frac{1}{c + \dots}}} \quad \text{by } C.$$

Then $C < \frac{1}{a}$, for a , the denominator of $\frac{1}{a}$, is smaller than the

denominator should be by $\frac{1}{b + \frac{1}{c + \dots}}$

the denominator of $\frac{1}{a}$ being smaller than it should be to give the true value of the fraction, and the numerator remaining

the same, the fraction is greater than it should be to give the true value, C . Again,

$$\frac{1}{a + \frac{1}{b}} < C,$$

for the denominator of $\frac{1}{b}$ is taken smaller than the denominator here should be, by

$$\frac{1}{c + \frac{1}{d + \dots}}$$

and $\frac{1}{b}$ is therefore greater than the real addition to a would be if all the integral fractions were taken. Hence,

$$a + \frac{1}{b} > \text{the true denominator,}$$

$$\text{And } \frac{1}{a + \frac{1}{b}} < \text{the true value of the fraction.}$$

Similarly,

$$\frac{1}{a + \frac{1}{b + \frac{1}{c}}} > C$$

for, c being too small, $\frac{1}{c}$ is too great, and therefore $b + \frac{1}{c}$ too great, whence

$$a + \frac{1}{b + \frac{1}{c}} \text{ is too small, and } \frac{1}{a + \frac{1}{b + \frac{1}{c}}} \text{ too great,}$$

to give the true value of the continued fraction. In a similar way it may be shown in any case that

Every convergent of an odd order is greater, and every convergent of an even order less, than the true value of the continued fraction. This value must therefore lie between the consecutive convergents.

178. To Deduce a Rule for obtaining directly a convergent of any order from the preceding ones.

By omitting all the integral fractions beyond a certain point, and finding the value of the resulting fraction as already indicated, we may obtain any approximating fraction, but it is often more convenient to obtain a convergent directly from its predecessors.

Examining the approximating fractions of the first, second, third, etc., orders, we see that

$$\frac{1}{a} = \frac{1}{a}$$

$$\frac{1}{a + \frac{1}{b}} = \frac{b}{ab+1}$$

$$\frac{1}{a + \frac{1}{b + \frac{1}{c}}} = \frac{b + \frac{1}{c}}{a(b + \frac{1}{c}) + 1} = \frac{bc+1}{(ab+1)c+a}$$

$$\frac{1}{a + \frac{1}{b + \frac{1}{c + \frac{1}{d}}}} = \frac{b(c + \frac{1}{d}) + 1}{(ab+1)(c + \frac{1}{d}) + a} = \frac{(bc+1)d+b}{[(ab+1)c+a]d+ab+1}$$

the numerator of the third approximating fraction may be obtained by multiplying the numerator of the second approximating fraction by the denominator of the third integral fraction and to this result adding the numerator of the first approximating fraction or convergent; the denominator of the third convergent may be obtained by multiplying the denominator of the second convergent by the denominator of the third integral fraction, and adding to this the denominator of the first convergent. Again, the fourth convergent may be formed from the third and second by the same law. To prove the law to be general, assume the $(n-1)th$ convergent, $\frac{T}{W}$, to be formed from the two pre-

ceeding by this law, $\frac{P}{Q}$ being the $(n-3)$ th and $\frac{R}{S}$ the $(n-2)$ th convergents. Then, k being the denominator of the $(n-1)$ th integral fraction, we have, by the assumed law,

$$\frac{T}{W} = \frac{Rk+P}{Sk+Q} \dots \dots \dots (1).$$

Now, $\frac{1}{n}$ representing the n th integral fraction and $\frac{T'}{W'}$ the n th convergent, $\frac{T'}{W'}$ may be obtained from $\frac{T}{W}$ by substituting therein for k , $k + \frac{1}{n}$, as this is the same as changing $\frac{1}{k}$ into

$$\frac{1}{k + \frac{1}{n}} \therefore \frac{T'}{W'} = \frac{R(k + \frac{1}{n}) + P}{S(k + \frac{1}{n}) + Q} = \frac{(Rk+P)n+R}{(Sk+Q)n+S} = \frac{Tn+R}{Wn+S} \text{ by (1),}$$

and this value of $\frac{T'}{W'}$ conforms to the assumed law. Hence

if this assumed law hold good for the formation of any convergent, it holds for the formation of the succeeding one. It has been shown that the law holds good for the formation of the fourth, hence it holds good for the formation of the fifth, and therefore for the sixth, and so on; in other words, it is general. Hence to obtain any convergent from the preceding, we have the

RULE.—Add the product of the numerator of the convergent last found, and the denominator of the next integral fraction, to the numerator of the preceding convergent, for the new numerator. Add the product of the denominator of the same convergent and the same multiplier, to the denominator of the preceding convergent, for the denominator.

179. Theorem.—The difference between any two consecutive convergents equals ± 1 divided by the product of the denominators.

The difference between the first and second is,

$$\frac{1}{a} - \frac{b}{ab+1} = \frac{ab+1-ab}{a(ab+1)} = \frac{1}{a(ab+1)}$$

The difference between the second and third is,

$$\frac{b}{ab+1} - \frac{bc+1}{(ab+1)c+a} = \frac{(ab+1)bc+ab-(ab+1)bc-ab-1}{(ab+1)[(ab+1)c+a]} = \frac{-1}{(ab+1)[(ab+1)c+a]}$$

$$\text{Again: } \frac{bc+1}{(ab+1)c+a} - \frac{(bc+1)d+b}{[(ab+1)c+a]d+ab+1} \\ = \frac{1}{[(ab+1)c+a]\{[(ab+1)c+a]d+ab+1\}}$$

In these cases the proposition is evidently true.

Now let $\frac{P}{Q}$, $\frac{R}{S}$, and $\frac{T}{W}$, be any three consecutive convergents. $\frac{P}{Q} - \frac{R}{S} = \frac{PS-QR}{QS}$ and $\frac{R}{S} - \frac{T}{W} = \frac{RW-ST}{SW}$, or since
(178) $T=Rk+P$ and $W=Sk+Q$,

$$\frac{R}{S} - \frac{T}{W} = \frac{(Sk+Q)R-(Rk+P)S}{SW} = \frac{QR-PS}{SW}$$

Since if $PS-QR=\pm 1$, $QR-PS=\mp 1$, we see that if the difference between any two successive convergents is formed in accordance with the proposition, the difference between the second of these convergents and its successor is formed in accordance with the same law. As the difference between the third and the fourth is formed in accordance with the law stated, the difference between the fourth and fifth is also formed in accordance therewith, and that being the case, the difference between the fifth and sixth is formed in the same way, and so on. Hence the proposition is always true.

180. When the minuend is a convergent of an odd order and the subtrahend a convergent of an even order the upper sign is used; in the opposite case the lower (177).

181. **Theorem.**—Any convergent expresses the true value of the continued fraction to within 1 divided by the square of the denominator of the convergent.

The difference between the continued fraction and any convergent is less than the difference between this convergent and the succeeding one, since (177) the true value of the continued fraction lies between them. Hence, preserving the notation of Art. 178, the $(n-1)$ th convergent is the value of the continued fraction to within $\frac{1}{WW'}$. Since $W' = Wn + S$, and a, b, c, n , etc., are all entire quantities, $W' > W \therefore W' < WW' \therefore \frac{1}{WW'} < \frac{1}{W^2}$. Since the $(n-1)$ th convergent is the value of the continued fraction to within $\frac{1}{WW'}$, it is the value thereof to within the greater quantity $\frac{1}{W^2}$. Q. E. D.

(D.) FRACTIONAL EXPONENTS AND NEGATIVE EXPONENTS.

I. FRACTIONAL EXPONENTS.

182. Lemma.—*The signification of a quantity with a fractional exponent is not changed by multiplying both terms of this fraction by the same quantity.*

Let us take the expression, $a^{\frac{2}{3}}$. This means that a is to be divided into 3 equal factors, of which 2 are to be taken and multiplied together; or that any one of the 3 equal factors into which a is divided is to be taken twice as a factor. Now if we divide each of these three equal factors into five equal factors we obtain the same factors in number and size as if a were divided into fifteen equal factors, five of these factors multiplied together being equal to one of the old factors. Hence, ten must be multiplied together to produce the same result as produced by multiplying together two of the old factors. In other words, if a is divided into fifteen equal factors, and ten of these factors are taken, the same result is obtained as was obtained by dividing a into three equal factors and taking two of these.

$a^{\frac{10}{15}}$ means that a is divided into fifteen equal factors, of which ten are taken. $a^{\frac{2}{3}}$ means that a is divided into three equal factors, of which two are taken. Hence, as just proved, $a^{\frac{2}{3}} = a^{\frac{10}{15}}$. $a^{\frac{10}{15}}$ can evidently be obtained from $a^{\frac{2}{3}}$ by simply multiplying both terms of the exponent of $a^{\frac{2}{3}}$ by 5. Hence, multiplying both terms of this exponent by 5 does not affect the value of the quantity signified by the expression $a^{\frac{2}{3}}$. In a similar manner we might reason on any fractional exponent.

Or, to make our discussion perfectly general, let us take $a^{\frac{m}{n}}$. This means that a is to be divided into n equal factors, of which m are taken. If, now, each of these factors be divided into s equal factors, it will evidently take s of these new factors, multiplied together, to equal one of the old ones. Since in a there are n factors of the old size, and in each of these s factors of the new size, there will be in a altogether ns factors of the new size. Since s new factors, multiplied together, equal one old factor, it will take ms new factors, multiplied together, to equal m of the old factors. If, then, we take ms of the new factors of a , and multiply them together, we obtain the same result as if we take m of the old factors and multiply them together. Hence, $a^{\frac{m}{n}} = a^{\frac{ms}{ns}}$. Since these letters may represent any quantities whatever, we conclude, generally, that the significance of a quantity with a fractional exponent is not changed by multiplying both terms of the fractional exponent by the same quantity.

183. Corollary.—*The significance of any quantity with a fractional exponent is not altered by dividing both terms of this exponent by the same quantity.*

This is an obvious consequence of the main proposition; if $a^{\frac{m}{n}} = a^{\frac{ms}{ns}}$, then $a^{\frac{ms}{ns}} = a^{\frac{m}{n}}$.

To Deduce a Rule for multiplying together quantities having fractional exponents.

184. I.—Law of Exponents.

By (155) we can always transform the given fractional exponents so that the exponents may have the same fractional unit.

Let it be required, then, to multiply $a^{\frac{2}{7}}$ by $a^{\frac{3}{7}}$. This means that a is to be divided into seven equal factors, and one of these factors ($a^{\frac{1}{7}}$) is to be taken first twice, and then three times more, as a factor. If we take a factor first twice, and then three times, we take it five times. Hence the result obtained by taking one of the seven equal factors of a first twice, and then three times, as a factor is the same as would have been obtained by taking it five times as a factor. We indicate (17) that a is to be divided into seven equal factors, of which one is taken five times as a factor, or, which is the same thing, of which five are multiplied together, thus $a^{\frac{5}{7}}$. Hence, $a^{\frac{2}{7}} \times a^{\frac{3}{7}} = a^{\frac{5}{7}}$. In a similar manner we may reason on the multiplication of any two quantities affected with fractional exponents.

Or, to make the discussion perfectly general, let us multiply $a^{\frac{m}{n}}$ by $a^{\frac{s}{n}}$. This means that a is to be divided into n equal factors, and the product of m of these is to be multiplied by the product of s of them; or, which is the same thing, any one of them is to be taken, first m times, and then s times, as a factor. If we take any one of these equal factors ($a^{\frac{1}{n}}$), first m times and then s times, as a factor, we evidently take it $m+s$ times altogether. We indicate that a is to be divided into n equal factors, of which one is to be taken $m+s$ times as a factor; or (which amounts to the same), of which $m+s$ are to be multiplied together, thus: $a^{\frac{m+s}{n}}$. As the result of taking one of these equal factors

$(\frac{1}{a^n})$, first m times, and then s times, is the same as the result obtained by taking one of them $m+s$ times as a factor, the product of $a^{\frac{m}{n}}$ and $a^{\frac{s}{n}}$ is $a^{\frac{m+s}{n}}$.

A review of the work shows that the result of the multiplication of $a^{\frac{m}{n}}$ by $a^{\frac{s}{n}}$ might have been written at once by writing a in the product with an exponent equal to the sum of its exponents in the factors given.

II. The law of Coefficients and of Signs of Art. 62 will evidently not be changed by the entrance into either the multiplicand or multiplier, or both, of quantities having fractional exponents.

Thus, let us multiply $3mx^{\frac{a}{c}}y^{\frac{e}{d}}$ by $-5nx^{\frac{i}{c}}y^{\frac{b}{d}}$. This product may be indicated thus:

$$3mx^{\frac{a}{c}}y^{\frac{e}{d}} \times -5nx^{\frac{i}{c}}y^{\frac{b}{d}} \text{ or (61) } -3 \times 5mnx^{\frac{a}{c}}x^{\frac{i}{c}}y^{\frac{e}{d}}y^{\frac{b}{d}},$$

or by (I) $-15mnx^{\frac{a+i}{c}}y^{\frac{e+b}{d}}$.

Reviewing the whole discussion, recollecting that the letters employed may represent any quantities whatever, we deduce, for multiplying factors containing quantities with fractional exponents, the

RULE.—*Multiply the coefficients together for a new coefficient. After this write all the letters entering the original factors, giving to each an exponent equal to the sum of its exponents in the two factors. Prefix the minus sign to the product if the signs of the factors are unlike. Prefix no sign (+ being understood), if the signs of the factors are alike.*

185. Scholium.—FROM THE FOREGOING DISCUSSION, AND THE DEMONSTRATIONS IN CHAPTER II., SEC. III., IT FOLLOWS THAT THE RULES THEREIN GIVEN FOR MULTIPLYING POLYNOMIALS ARE APPLICABLE WITHOUT CHANGE, WHEN ANY OR ALL OF THE TERMS OF THE POLYNOMIALS HAVE LETTERS WITH FRACTIONAL EXPONENTS.

EXAMPLES.

Multiply together:

1. $32a^{\frac{1}{2}}b^{\frac{1}{3}}c^{\frac{1}{4}}$, $2a^{\frac{2}{3}}b^{\frac{1}{2}}c^{\frac{1}{4}}$, and $5a^{\frac{4}{3}}b^{\frac{2}{3}}c^{\frac{1}{4}}$.

2. $5a^{\frac{m}{n}}b^{\frac{1}{c}}c^{\frac{1}{s}}$, $3a^{\frac{2m}{n}}b^{\frac{2}{c}}c^{\frac{2}{s}}$, and $6a^{\frac{m}{n}}b^{\frac{r}{c}}c^{\frac{r}{s}}$. *Ans.* $90a^{\frac{4m}{n}}b^{\frac{r}{c}}c^{\frac{4+r}{s}}$.

3. $2a(x+y)^{\frac{s}{t}}$, $5c(x+y)^{\frac{r}{m}}$, and $3x^{\frac{1}{2}}(x+y)^{\frac{1}{ms}}$.
Ans. $30acx^{\frac{1}{2}}(x+y)^{\frac{ms+rt+t+1}{mt}}$.

4. $4a(c+d)^{\frac{2}{3}}$, $8x^{\frac{1}{2}}(c+d)^{\frac{5}{6}}$ and $4x^{\frac{3}{2}}(c+d)^{\frac{1}{3}}$.
Ans. $128ax^{\frac{7}{2}}(c+d)^2$.

5. $7c^{\frac{1}{2}}d^{\frac{2}{3}}(x+y)^{\frac{2}{5}}$, $8c^{\frac{2}{3}}d^{\frac{2}{3}}(x+y)^{\frac{4}{5}}$ and $9c^{\frac{2}{3}}d^{\frac{4}{5}}(x+y)^{\frac{6}{5}}$.
Ans. $504c^{\frac{2}{3}}d^{\frac{2}{3}}(x+y)^{\frac{14}{5}}$.

6. $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{2}} + a^{\frac{7}{3}}b - a^{\frac{1}{3}}b^{\frac{3}{2}} + a^{\frac{5}{3}}b^2 - a^{\frac{1}{3}}b^{\frac{5}{2}} + a^{\frac{2}{3}}b^3 - ab^{\frac{7}{2}} + a^{\frac{1}{3}}b^4 - b^{\frac{9}{2}}$
 $\quad \quad \quad a^{\frac{1}{2}} + b^{\frac{1}{2}}$

 $a^{\frac{5}{2}} - a^{\frac{2}{3}}b^{\frac{1}{2}} + a^{\frac{1}{3}}b - a^{\frac{7}{3}}b^{\frac{3}{2}} + a^{\frac{1}{3}}b^2 - a^{\frac{5}{3}}b^{\frac{5}{2}} + a^{\frac{2}{3}}b^3 - a^{\frac{1}{3}}b^{\frac{7}{2}} + ab^4 - a^{\frac{1}{3}}b^{\frac{9}{2}}$
 $\quad \quad \quad + a^{\frac{2}{3}}b^{\frac{1}{2}} - a^{\frac{1}{3}}b + a^{\frac{7}{3}}b^{\frac{3}{2}} - a^{\frac{1}{3}}b^2 + a^{\frac{5}{3}}b^{\frac{5}{2}} - a^{\frac{2}{3}}b^3 + a^{\frac{1}{3}}b^{\frac{7}{2}} - ab^4 + a^{\frac{1}{3}}b^{\frac{9}{2}} - b^{\frac{5}{2}}$

 $a^{\frac{5}{2}} \quad \quad \quad - b^{\frac{5}{2}}$

7. Multiply

$5a^{\frac{1}{2}}b^{\frac{1}{3}} - 6a^{\frac{1}{3}}b^{\frac{1}{2}} + 4a^{\frac{1}{2}}b^{\frac{1}{3}} - 7a^{\frac{1}{3}}b^{\frac{1}{2}}$ by $2a^{\frac{1}{2}}b^{\frac{5}{3}} - ab^{\frac{1}{2}} + 2b^{\frac{2}{3}}$.

8. Multiply $5a^{\frac{2}{3}} - 2a^{\frac{1}{3}}b^{\frac{1}{2}} + 4a^{\frac{1}{2}}b^{\frac{1}{3}}$ by $a^{\frac{1}{2}} - 4a^{\frac{1}{3}}b^{\frac{1}{2}} + 2b^{\frac{1}{3}}$.

Ans. $5a^{\frac{7}{6}} - 22a^{\frac{2}{3}}b^{\frac{1}{2}} + 12a^{\frac{1}{2}}b^{\frac{1}{3}} - 6a^{\frac{1}{3}}b^{\frac{1}{2}} - 4a^{\frac{1}{2}}b^{\frac{2}{3}} + 8a^{\frac{1}{3}}b^{\frac{1}{3}}$.

9. Multiply $x^{\frac{3}{2}} - 4x^{\frac{1}{2}} + 2$ by $x - x^{\frac{1}{2}}$.

Ans. $x^{\frac{5}{2}} - x^2 - 4x^{\frac{3}{2}} + 6x - 2x^{\frac{1}{2}}$.

10. Multiply $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y^{\frac{3}{2}}$ by $x^2 - xy + y^2$.

186. To Deduce a Rule for division, when either the dividend or the divisor contains quantities having fractional exponents.

In division, as has already been stated, it is required to find a quantity which, multiplied by one of two given quantities, will produce the other.

I. Law of Exponents.

As before, we assume that all fractional exponents of each letter have been transformed so as to have a common denominator. Since the exponent of any letter in the quotient, must, on being added to the exponent of the same letter in the divisor, give the exponent of that letter in the dividend, the exponent of any letter in the quotient must be the difference between the exponent of the same letter in the dividend and the exponent of that letter in the divisor. Hence the law of exponents in division is the same, whether the exponents be entire or fractional.

II. Laws of Signs and Coefficients.

Division being the converse of multiplication, the laws of coefficients and signs will evidently not be changed on account of the entrance into dividend or divisor of quantities with fractional exponents.

From the foregoing considerations we deduce the following

RULE.—*Divide the dividend by the divisor in precisely the same manner as when the exponents in the dividend and divisor are all integral.*

187. Scholium.—FROM THIS DISCUSSION AND THE DEMONSTRATIONS IN CHAPTER II., SECTION IV., IT WILL BE SEEN THAT THE RULES THEREIN GIVEN, FOR THE DIVISION OF POLYNOMIALS, ARE APPLICABLE WITHOUT MODIFICATION, WHEN ANY OR ALL OF THE TERMS OF THE POLYNOMIALS CONTAIN LETTERS WITH FRACTIONAL EXPONENTS.

EXAMPLES.

1. Divide $154a^{\frac{3}{2}}b^{\frac{m}{2}}(x+y)^{\frac{1}{2}}$ by $77a^{\frac{1}{2}}b^{\frac{1}{2}}(x+y)^{\frac{1}{2}}$.

Ans. $2a^{\frac{1}{2}}b^{\frac{m-1}{2}}(x+y)^{\frac{1}{2}}$.

2. Divide $86a^{\frac{2}{3}}b^{\frac{5}{3}}c$ by $12a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}}$.

Ans. $3a^{\frac{1}{3}}b^{\frac{4}{3}}c^{\frac{2}{3}}$.

3. Divide $92(x+y)^{\frac{2}{3}}(c+d)^{\frac{5}{3}}$ by $46(x+y)^{\frac{1}{3}}(c+d)^{\frac{2}{3}}$.

4. Divide $\frac{1}{5}x^{\frac{8}{5}} - \frac{3}{10}x^{\frac{7}{5}} + \frac{1}{2}x^{\frac{6}{5}} - \frac{1}{3}x^{\frac{5}{5}} - \frac{3}{20}x^{\frac{4}{5}} + \frac{1}{4}x^{\frac{3}{5}}$ by $x^{\frac{1}{5}} - \frac{3}{5}x^{\frac{4}{5}}$.

Ans. $\frac{1}{2}x^{\frac{1}{5}} - \frac{1}{3}x^{\frac{4}{5}} + \frac{1}{4}x^{\frac{1}{5}}$.

5. Divide $x^3 + 2x^2y + x^{\frac{5}{2}}y^{\frac{1}{2}} + x^{\frac{3}{2}}y^{\frac{3}{2}} + 2xy^2 + x^{\frac{1}{2}}y^{\frac{5}{2}} + y^3$ by $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$.

Ans. $x^2 + xy + y^2$.

6. Divide $a^{\frac{7}{2}} - b^{\frac{7}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

Ans. $a^3 + a^{\frac{5}{2}}b^{\frac{1}{2}} + a^2b + a^{\frac{3}{2}}b^{\frac{3}{2}} + ab^2 + a^{\frac{1}{2}}b^{\frac{5}{2}} + b^3$.

7. Divide $100a^{\frac{1}{2}} - 440a^{\frac{3}{2}}k^{\frac{1}{10}} + 235a^{\frac{5}{2}}k^{\frac{3}{10}} - 30a^{\frac{7}{2}}k^{\frac{5}{10}}$ by $5a^{\frac{1}{2}} - 2a^{\frac{1}{2}}k^{\frac{1}{10}}$.

Ans. $20a^{\frac{1}{2}} - 80a^{\frac{1}{10}}k^{\frac{1}{10}} + 15k^{\frac{1}{5}}$.

8. Divide $48x^{\frac{3}{2}} - 76a^{\frac{1}{2}}x^{\frac{5}{2}} - 64a^{\frac{3}{2}}x^{\frac{7}{2}} + 105a^{\frac{5}{2}}x^{\frac{9}{2}}$ by $2x^{\frac{1}{2}} - 3a^{\frac{1}{2}}x^{\frac{3}{2}}$.

Ans. $24x^{\frac{1}{2}} - 2a^{\frac{1}{2}}x^{\frac{3}{2}} - 35a^{\frac{3}{2}}$.

II. NEGATIVE EXPONENTS.

188. To Deduce Rules for the multiplication of quantities affected by negative exponents.

The laws of coefficients and signs being evidently not changed by the differences in the exponents, the only law that must be explained here in detail is the law of exponents.

Let it be required, then, to multiply a^{-m} by a^{-n} ; $a^{-m} = \frac{1}{a^m}$ (86),

$a^{-n} = \frac{1}{a^n}$. Hence, $a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} = \frac{1}{a^{m+n}}$; $\frac{1}{a^{m+n}} = a^{-(m+n)} = a^{-m-n}$. Here we see that the exponent of a in the product is the sum of the exponents of a in the two factors

given. Since the letters employed may represent any quantities whatever, the exponent of the product of any quantity having a negative exponent, and this quantity with another negative exponent, is always the sum of the exponents in the two factors. In a similar manner we can show that $a^{-m} \times a^n$ is a^{-m+n} ; i. e., if the exponent of a quantity is positive in one factor and negative in the other, the exponent of this quantity in the product will still be the algebraic sum of the exponents of the quantity in the two factors. Hence the law of exponents in multiplication holds good for the multiplication of quantities having negative exponents as well as for the multiplication of those having positive exponents. Hence the rules already deduced in multiplication require no change in being applied to factors containing quantities with negative exponents.

189. To Deduce Rules for the division of quantities having negative exponents.

Since division is the converse of multiplication, and since even when one or both of the factors contain quantities with negative exponents, the exponent of any letter in the product (or dividend) is the sum of the exponents of the same letter in the multiplicand (or divisor) and the multiplier (or quotient); the Law of Exponents in division is the same when the dividend or divisor, or both, contain quantities with negative exponents, as when they contain only quantities with positive exponents. The laws of coefficients and signs being evidently the same whether the quantities have positive or negative exponents, we see that the rules for division, Articles 79-89, are equally applicable, whether the dividend or divisor, or both, contain only terms with positive or with negative exponents, or contain terms involving both positive and negative exponents.

EXAMPLES.

1. Multiply $4a^{\frac{1}{2}}b^{-\frac{1}{2}}c^4 - 3a^2b^{-\frac{7}{2}}d^{-3}$ by $3a^{\frac{2}{3}}b^{-\frac{1}{2}} + 7a^{-\frac{1}{2}}b^{\frac{3}{2}}$.
 Ans. $4c^4(3a^{\frac{1}{2}}b^{-\frac{5}{2}} + 7a^{-\frac{1}{2}}b^{\frac{4}{2}}) - 3d^{-3}(3a^{\frac{1}{2}}b^{-\frac{1}{2}} + 7a^{\frac{1}{2}}b^{-\frac{1}{2}})$.

2. Multiply $x^{-2p} + 3a^m x^{-2p} - 10a^{2m} x^{-2p}$ by $a^2 x^2 + 5a^{m+2} x^{2+p} - 2a^{2m+2} x^{2+2p}$.

Ans. $a^2 x^{2-2p} + 8a^{m+2} x^{2-2p} + 3a^{2m+2} x^{2-2p} - 56a^{3m+2} x^{2+2p} + 20a^{4m+2} x^{2+2p}$

3. Divide $x^{\frac{1}{2}} y^{-\frac{1}{2}} + 2 + x^{-\frac{1}{2}} y^{\frac{1}{2}}$ by $x^{-\frac{1}{2}} + y^{-\frac{1}{2}}$. Ans. $x^{\frac{1}{2}} + y^{\frac{1}{2}}$.

4. Divide $x^2 - y^2$ by $x^{\frac{5}{2}} + x^2 y^{\frac{1}{2}} + x^{\frac{3}{2}} y + xy^{\frac{3}{2}} + x^{\frac{1}{2}} y^2 + y^{\frac{5}{2}}$.

Verify.

5. Divide $x^{-1} - 2x^{-\frac{1}{2}} + 1$ by $x^{-\frac{1}{2}} - 2x^{-\frac{1}{6}} + 1$.

Ans. $x^{-\frac{2}{3}} + 2x^{-\frac{1}{2}} + 3x^{-\frac{1}{3}} + 2x^{-\frac{1}{6}} + 1$.

6. Divide $x^{-\frac{5}{2}} - 4x^{-\frac{3}{2}} + 6x^{-1} - x^{-2} - 2x^{-\frac{1}{2}}$ by $x^{-\frac{3}{2}} - 4x^{-\frac{1}{2}} + 2$.

Ans. $x^{-1} - x^{-\frac{1}{2}}$.

7. Find the highest common divisor of

$3x^{-5m} - 10x^{-3m} + 15x^{-m} + 8$, and $x^{-5m} - 2x^{-4m} - 6x^{-3m} + 4x^{-2m} + 13x^{-m} + 6$.

Ans. $x^{-3m} + 3x^{-2m} + 3x^{-m} + 1$.

8. Find the lowest common multiple of

$x^{-3a} + 11x^{-a} - 6x^{-2a} - 6$, $x^{-3a} - 9x^{-2a} + 26x^{-a} - 24$, and $x^{-3a} - 8x^{-2a} + 19x^{-a} - 12$.

Ans. $(x^{-a} - 1)(x^{-a} - 2)(x^{-a} - 3)(x^{-a} - 4)$

It will be observed that, in these examples, the terms *highest* and *lowest* are to be considered as applying to the exponents, without regard to their signs. This is the case whenever we have expressions of which the terms contain quantities with negative exponents.

190. Important General Scholium.—*The rules for multiplication and division are applicable WITHOUT ANY RESERVATION, whether the quantities considered have exponents positive or negative, entire or fractional. Hence all principles, in so far as they depend on these rules for demonstration, are true, whether the exponents be positive or negative, entire or fractional.*

CHAPTER V.

INVOLUTION AND EVOLUTION.

SECTION I.

INVOLUTION.

191. We have found (8) that when a product is formed by multiplying together two or more equal factors, this product is called a **Power** of any one of the equal factors, each factor being a **Root of the Power**. The process of multiplication is then called **Involution**. We are said to *raise* any one of the equal factors *to the power* resulting. See Article 8.

192. **Involution**, then, is the process of raising a quantity to a certain power, i. e., of taking the given quantity a certain number of times as a factor.

193. **The Degree** of a power is the number of equal factors taken, i. e., the number of times the given quantity is taken as a factor.

The *degree* of a power gives the power its name. Thus a power formed by the multiplication together of *two* equal factors, is of the *second degree*, and is the *second power* of either factor. If a quantity be taken *three* times as a factor, we have the *third power* or *cube* of the quantity; if *four* times, the power is of the *fourth degree*, or is the *fourth power* of the quantity; and so on.

194. **An Exponent** is a small symbol written above and a little to the right of a quantity; it indicates:

1°. *If positive and entire.*

The number of times the quantity is to be taken as a factor. Thus a^2 indicates that a is to be taken twice as a

factor. We should carefully note the difference in meaning between a *Coefficient* and an *Exponent*. A *coefficient* indicates *continued addition*; an *exponent*, *continued multiplication*. Thus, $3a = a + a + a$, while $a^3 = a . a . a$.

2°. *If positive and fractional.*

If the exponent be a positive fraction, the numerator of which is 1, it indicates that the quantity affected with it is to be divided into as many equal factors as there are units in the denominator of the exponent, and that one of these factors is to be taken or considered. If the numerator of the fraction be greater than 1, the exponent indicates that the quantity affected therewith is to be divided into as many equal factors as there are units in the denominator, and as many of these factors are to be taken as there are units in the numerator. Thus $a^{\frac{1}{3}}$ indicates that a is to be divided into three equal factors, one of which is to be taken; $a^{\frac{3}{5}}$ indicates that a is to be divided into five equal factors, the product of three of which is to be taken.

3°. *If negative.*

A quantity having a negative exponent is the same as the reciprocal of the quantity, with a numerically equal positive exponent. Thus $a^{-2} = \frac{1}{a^2}$; $x^{-\frac{5}{6}} = \frac{1}{x^{\frac{5}{6}}}$.

$$a^{-2} = \frac{1}{a^2}; x^{-\frac{5}{6}} = \frac{1}{x^{\frac{5}{6}}}$$

NOTE.—As exponents do not always indicate that the quantities affected by them are to be raised to powers merely, it is not always exactly correct to speak of raising a quantity to the power indicated by a given exponent. At the same time as this form of expression has passed into general use, and as it is not likely to mislead the careful student we shall use it here, in the belief that it is better *ceteris paribus* to use intelligently an expression sanctioned by long usage, than to create confusion by attempting to replace it.

When no exponent is written, 1 is understood.

INVOLUTION OF MONOMIALS.

195. To Deduce a Rule for raising a Monomial to any required power.

10. When the exponent of the required power is positive and integral.

Let it be required to raise $3a^2b^3c^4$ to the fourth power. By definition, this power is the product obtained by taking $3a^2b^3c^4$ four times as a factor, i. e., by multiplying it by itself three times, thus:

$3a^2b^3c^4 \times 3a^2b^3c^4 \times 3a^2b^3c^4 \times 3a^2b^3c^4 = 3.3.3.3a^2a^2a^2a^2b^3b^3b^3b^3c^4c^4c^4c^4$,
or by (62) $81a^{16}b^{12}c^{16}$. This result might evidently have been obtained at once, by raising the coefficient 3 to the fourth power for the new coefficient, and after this writing the literal part of the given monomial, giving to each letter an exponent equal to the exponent of that letter in the original monomial, multiplied by the exponent of the required power. If now we should multiply $3a^2b^3c^4$ by itself four times, that is, raise it to the fifth power, we should have (62) $243a^{10}b^{15}c^{20}$, which, we see at once, might have been obtained directly in the way just pointed out.

To show that this method of forming a power of a monomial is *general in its applicability*, suppose this method applicable to the formation of a power, the exponent of which is m . Then the m th power of $3a^2b^3c^4$ will be $3^ma^{2m}b^{3m}c^{4m}$. If now we multiply $3^ma^{2m}b^{3m}c^{4m}$ by $3a^2b^3c^4$, we obtain (62) $3^{m+1}a^{2m+2}b^{3m+3}c^{4m+4}$, which is equal to

$$3^{m+1}a^{2(m+1)}b^{3(m+1)}c^{4(m+1)}.$$

On comparing the result with the given quantity, we see that the coefficient of the result is the $(m+1)$ th power of the given coefficient, that each letter of the given monomial enters the result; that the exponent of every letter in the result equals the exponent of the same letter in the original quantity multiplied by the exponent of the power, $m+1$. Hence, if the m th power of a monomial be formed by the law given above, the $(m+1)$ th power will be formed by the same law. In other words, if any power of a monomial be formed in accordance with this law, the next higher power will also be formed in accordance therewith. Hence as this law holds good for the fifth power, it holds good also

for the sixth, and therefore for the seventh, and so on, i. e., it is general.

2°. The exponent of the required power is a positive fraction

Let $\frac{p}{q}$ represent this exponent, p and q being both positive and integral, and let it be required to raise ab^mc^r to the power $\frac{p}{q}$. This means that ab^mc^r is to be separated into q equal factors, of which p are to be taken, and multiplied together. We may evidently separate ab^mc^r into q equal factors by taking a , b^m and c^r , separating each of them into q equal factors and taking the product of the results. One of the equal factors into which a is separated is represented thus: $a^{\frac{1}{q}}$; one of those into which b^m is separated, is represented thus: $b^{\frac{m}{q}}$; and one of those into which c^r is separated is represented thus: $c^{\frac{r}{q}}$. Hence $a^{\frac{1}{q}}b^{\frac{m}{q}}c^{\frac{r}{q}}$ represents one of the q factors into which ab^mc^r is divided. Taking the product of p of these is the same as raising any one to the p th power; hence if we raise $a^{\frac{1}{q}}b^{\frac{m}{q}}c^{\frac{r}{q}}$ to the p th power we shall obtain $(ab^mc^r)^{\frac{p}{q}}$. Raising $a^{\frac{1}{q}}b^{\frac{m}{q}}c^{\frac{r}{q}}$ to the p th power, by 1°, since p is positive and entire, we obtain $a^{\frac{p}{q}}b^{\frac{mp}{q}}c^{\frac{rp}{q}}$, which evidently conforms to the law of formation indicated in 1°, and might have been obtained directly by the method there laid down.

As every positive fraction may be reduced to the form $\frac{p}{q}$ in which both p and q are positive integers, this discussion covers every case in which the exponent of the required power is a positive fraction.

3°. The exponent of the required power is negative, and either entire or fractional.

Let it be required to find $(ab^m c^n)^{-r}$. By (86).

$$(ab^m c^n)^{-r} = \frac{1}{(ab^m c^n)^r} = \frac{1}{a^r b^{mr} c^{nr}} = a^{-r} b^{-mr} c^{-nr} \quad (86),$$

This result might evidently have been obtained directly in the manner indicated in 1^o, as it conforms to the law there given.

From the preceding discussion we deduce the following general

RULE.—*Perform on the numerical coefficient the operations indicated by the exponent of the required power; after the result write the literal part, giving to each letter an exponent equal to the original exponent multiplied by the exponent of the required power.*

196. Corollary I.—**A Fraction may be raised to any power, by the following**

RULE.—*Raise the numerator to the required power for a new numerator and the denominator to the same power, for a new denominator.*

For, let it be required to find $\left(\frac{a}{b}\right)^m$, $\frac{a}{b}$ representing any fraction whatever, and m any exponent, positive or negative, entire or fractional. By (86)

$$\frac{a}{b} = ab^{-1} \text{ and } (ab^{-1})^m = a^m b^{-m} \quad (194) \text{ or } \frac{a^m}{b^m}.$$

197. Corollary II.—**A Mixed Quantity may be raised to any power by the following**

RULE.—*Reduce the mixed quantity to an equivalent fraction; raise this fraction to the required power, and reduce, if desired, the result to a mixed quantity.*

The truth of this proposition is evident from the fact that the required change of form does not alter the value of the mixed quantity.

198. Law of Signs in Involution.—**ALL powers of positive quantities are positive; EVEN powers of negative quan-**

ties are positive, and odd powers of negative quantities are negative.

For (62) the continued product of any number of positive quantities is positive; again, the product of any two negative quantities being positive, the continued product of any number of times two negative quantities is positive. Thus, if $2n$ negative quantities are to be multiplied together, as the product of any two is positive, the product of n times two will be the continued product of n positive factors, and therefore positive. Finally, the product of an odd number of negative quantities is always obtained by multiplying the product of an even number of negative quantities (which has just been shown to be positive) by a negative quantity, and is therefore negative.

EXAMPLES.

1.

$$\left(9a^{\frac{5}{2}}b^{\frac{3}{2}}c^{\frac{1}{2}}\right)^2 = ? \quad \left(2a^{-\frac{1}{2}}b^{-\frac{3}{2}}c^{\frac{1}{2}}\right)^6 = ? \quad \left\{-3a^{\frac{1}{2}}b^{-\frac{5}{2}}[x+y]^{\frac{r}{2}}e^2f\right\}^4 = ?$$

2.

$$\left\{5x^m y^{-\frac{r}{2}}(c+d)^{\frac{p}{2}}z^s\right\}^7 = ? \quad \left\{\pm 16a^{-\frac{3}{2}}b^{\frac{5}{2}}(x+y+z)^{-\frac{m}{2}}wx^2z^m\right\}^2 = ?$$

3.

$$\left\{\pm r s x^{-\frac{a}{2}}(c+d)^{\frac{p}{2}}z^{2m+1}\right\}^3 = ? \quad \left\{\mp abc(x+y+z)^{-\frac{2m+1}{2}}w^2z^2\right\}^{11} = ?$$

4.

$$\left\{\pm ab(s^2 - x^2 + y^2)^{\frac{2}{c}+3}(d+e)^{-\frac{m}{2}}w^2s^2v^{\frac{m+1}{c}}\right\}^{13} = ?$$

5.

$$\left\{\frac{ab(x^2 - y^2)^3}{cdx(s^2 + a^2)^4}\right\}^a = ? \quad \left\{\frac{a^m d^2(r+s)^{\frac{m}{2}}a}{xy(s^2 - y^2)^c}\right\}^r = ? \quad \left\{a + c \frac{mx^2s}{yrs}\right\}^{10} = ?$$

INVOLUTION OF BINOMIALS.

199. From the definition of Involution it follows that a binomial may be raised to any power by multiplying it by

itself the required number of times. The process of involving binomials is, however, very much shortened by the employment of the **Binomial Formula**.

200. The Binomial Formula is a formula, by the use of which a binomial may be raised to any power without the actual performance of the indicated continued multiplication. This formula was discovered and announced, but not proved, by Sir Isaac Newton.

201. To Deduce the Binomial Formula.

1°. The exponent of the required power is positive and integral.

The following powers of $x+y$ are formed by actual multiplication:

$$(x+y)^1 = x+y.$$

$$(x+y)^2 = x^2 + 2xy + y^2.$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

An examination of these powers reveals the fact that they all conform to the following laws, viz:

I. Law of Exponents.—The exponent of the leading letter is, in the first term, equal to the exponent of the required power, and diminishes by 1 in each term toward the right, becoming 0 in the last term; the exponent of the other letter is 0 in the first term, and increases by 1 in each term toward the right, becoming, in the last term, equal to the exponent of the required power.

II. Law of Coefficients.—The coefficient of the first term is unity; the coefficient of any term after the first may be formed by multiplying the coefficient of the preceding term by the exponent of the leading letter in that term, and

dividing that product by the number of terms preceding the one of which the coefficient is required.

From the nature of the case, we see that the same laws will probably hold good in the formation of any power whatever. To prove that they must hold good for the formation of any power, it will be sufficient to show that if they hold good for any power they must hold good for the next higher power. Assume them true for the m th power. Then

$$(x+y)^m = x^m + mx^{m-1}y + m \frac{m-1}{1.2} x^{m-2}y^2 + m \frac{m-1}{2} \frac{m-2}{3} x^{m-3}y^3 + \dots + x+y$$

$$x^{m+1} + mx^my + m \frac{m-1}{1.2} x^{m-1}y^2 + m \frac{m-1}{2} \frac{m-2}{3} x^{m-2}y^3 + \dots + x^my + mx^{m-1}y^2 + m \frac{m-1}{2} x^{m-2}y^3 + \dots$$

$$(x+y)^{m+1} =$$

$$x^{m+1} + m \left| x^my + m \frac{m-1}{2} x^{m-1}y^2 + m \frac{m-1}{2} \frac{m-2}{3} x^{m-2}y^3 + \dots \right| + 1 \left| \begin{array}{c} +m \\ +m \frac{m-1}{2} \end{array} \right|$$

$$\begin{aligned} \text{Now, } m \frac{m-1}{2} + m &= m \left(\frac{m-1}{2} + 1 \right) = m \left(\frac{m-1+2}{2} \right) = \frac{(m+1)m}{2} \\ m \frac{m-1}{2} \frac{m-2}{3} + m \frac{m-1}{2} &= m \frac{m-1}{2} \left(\frac{m-2}{3} + 1 \right) = \\ m \frac{m-1}{2} \left(\frac{m-2+3}{3} \right) &= m \frac{m-1}{2} \frac{m+1}{3} = (m+1) \frac{m}{2} \frac{m-1}{3} \end{aligned}$$

Substituting in the product of $(x+y)^m$ and $(x+y)$,

$$(x+y)^{m+1} = x^{m+1} + (m+1)x^my + (m+1) \frac{m}{2} x^{m-1}y^2 + (m+1) \frac{m(m-1)}{1.2.3} x^{m-2}y^3 + \dots + y^{m+1} \quad (3)$$

which evidently conforms to the laws I., II. Hence, as the $(m+1)$ th power is the next power higher than the m th:

Proposition I.—If the laws I., II., hold good for the for-

mation of any power, they must hold good for the formation of the next higher power.

As they hold good for the powers from the first to the fifth inclusive, they must hold good for the sixth, and therefore for the seventh, the eighth, and so on for all powers with positive integral exponents.

2°. The exponent of the required power is a positive fraction.

As any algebraic expression in which the symbol representing any quantity occurs is a function of that quantity, a polynomial formed according to laws I., II., m being the coefficient letter, is a function of m , and may be denoted by $f(m)$; similarly, if n be the coefficient letter, the polynomial may be denoted by $f(n)$. In what follows, the functional symbol, f , will always have this meaning, viz., that the polynomial denoted is formed according to laws I., II., the symbol in the parenthesis indicating the coefficient of the second term, and the quantity, therefore, from which all the succeeding coefficients are derived. It is clear that $f(m).f(n)=f(m+n)$, if m and n be positive integers, since in this case $f(m)=(1+x)^m$, and $f(n)=(1+x)^n$, so that $f(m).f(n)=(1+x)^m.(1+x)^n=(1+x)^{m+n}=f(m+n)$. Now the *form* of the product of $f(m)$ and $f(n)$ cannot be affected by changes in m and n , although its value may be altered. Hence $f(m).f(n)$ is *always* $f(m+n)$.

This may be rendered clearer by direct multiplication; thus:

Suppose that

$$f(m)=1+mx+M'x^2+M''x^3+\dots \quad (1)$$

$$f(n)=1+nx+N'x^2+N''x^3+\dots \quad (2)$$

m and n being any quantities whatever, and M' , M'' , $\dots$, N' , N'' , $\dots$ coefficients formed by law II., p. 123.

Multiplying (1) by (2), member by member, we obtain,

$$f(m).f(n)=1+m\left| \begin{array}{c} x+M' \\ +n \\ +mn \end{array} \right| \left| \begin{array}{c} x^2+M'' \\ +N' \\ +M'n \end{array} \right| x^3+\dots \quad (3)$$

$$\left. \begin{array}{l} \text{Now, as } M' = \frac{m(m-1)}{2} \\ \text{and } N' = \frac{n(n-1)}{2} \end{array} \right\} \begin{aligned} M' + N' + mn &= \frac{m^2 - m + n^2 - n}{2} + mn, \\ &= \frac{m^2 + 2mn + n^2 - (m+n)}{2} \\ &= \frac{(m+n)^2 - (m+n)}{2} \\ &= \frac{(m+n)(m+n-1)}{2} \end{aligned}$$

$$\begin{aligned} &\text{Again,} \\ &M'' + N'' + M'n + mN' \\ &= m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} + n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} + m \cdot \frac{m-1}{2} \cdot n + \frac{n(n-1)}{2} \cdot m \\ &= \frac{m^3 + 2m - 3m^2}{6} + \frac{n^3 - 3n^2 + 2n}{6} + \frac{3m^2n - 3mn}{6} + \frac{3n^2m - 3mn}{6} \\ &= \frac{m^3 - 3m^2 + 3m^2n - 6mn + 2m + 3mn^2 + 2n - 3n^2 + n^3}{6} \\ &= \frac{(m^2 + 2mn + n^2 - 3m - 3n + 2)(m+n)}{6} \\ &= \frac{(m+n-2)(m+n-1)(m+n)}{3 \cdot 2 \cdot 1} \\ &= \frac{(m+n)(m+n-1)(m+n-2)}{1 \cdot 2 \cdot 3} \end{aligned}$$

In a similar manner we could show that the coefficient of x^4 is $\frac{(m+n)(m+n-1)(m+n-2)(m+n-3)}{1 \cdot 2 \cdot 3 \cdot 4}$, and so on for every coefficient. Hence the product, or $f(m) \cdot f(n)$, will be of the form,

$$1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 \cdot 2} x^2 + \frac{(m+n)(m+n-1)(m+n-2)}{1 \cdot 2 \cdot 3} x^3 + \dots \dots \dots (4).$$

The law of exponents is evidently the same as in (1^o) and, as just shown, the law of coefficients is also the same. The product, then, will be of the same form as either factor, $(m+n)$ occurring wherever m occurs in the first or n in the second, i. e., the product will be $f(m+n)$,

Let, now, $m=n$.

Then

$$f(m)=f(n), \text{ and } f(m).f(n)=f(m)^2=f(m+m)=f(2m).$$

If $n=2m$,

$$f(m).f(n)=f(m).f(2m)=f(m).[f(m)]^2=[f(m)]^3;$$

hence, as $f(m).f(n)=f(m+n)$,

$$[f(m)]^3=f(m+2m)=f(3m).$$

In a similar way we may show that

$$[f(m)]^4=f(4m);$$

$$[f(m)]^5=f(5m);$$

and that, in general, whatever the value of m ,

$$[f(m)]^q=f(qm) \quad (\text{i})$$

provided, only, that q be a positive integer.

Now let $m=\frac{r}{q}$, r and q being both positive integers.

Equation (i) becomes

$$\left[f\left(\frac{r}{q}\right)\right]^q=f\left(q\cdot\frac{r}{q}\right)=f(r) \quad (\text{ii})$$

Since, however, r is a positive integer, and we have established the binomial formula for powers with positive integral exponents, $f(r)=(1+x)^r$. Hence, from (ii),

$$(1+x)^r=\left[f\left(\frac{r}{q}\right)\right]^q \quad (\text{iii})$$

Extracting the q th root of each member,

$$(1+x)^{\frac{r}{q}}=f\left(\frac{r}{q}\right),$$

that is, with the meaning already assigned our symbols,

$$(1+x)^{\frac{r}{q}} = 1 + \frac{r}{q}x + \frac{R'}{Q}x^2 + \frac{R''}{Q''}x^3 + \dots \quad (5)$$

Since in (5) x may represent any quantity whatever,

$$\text{let } x = \frac{y}{z},$$

whence

$$1+x = 1 + \frac{y}{z} = \frac{z+y}{z} \therefore (1+x)^{\frac{r}{q}} = \frac{(z+y)^{\frac{r}{q}}}{z^{\frac{r}{q}}}.$$

Then

$$\frac{(z+y)^{\frac{r}{q}}}{z^{\frac{r}{q}}} = 1 + \frac{r}{q} \frac{y}{z} + \frac{R'}{Q} \frac{y^2}{z^2} + \frac{R''}{Q''} \frac{y^3}{z^3} + \dots \quad (6)$$

or, multiplying each of these equals by $z^{\frac{r}{q}}$,

$$(z+y)^{\frac{r}{q}} = z^{\frac{r}{q}} + \frac{r}{q} z^{\frac{r}{q}-1} y + \frac{R'}{Q} z^{\frac{r}{q}-2} y^2 + \frac{R''}{Q''} z^{\frac{r}{q}-3} y^3 + \dots \quad (7)$$

$\frac{R'}{Q}, \frac{R''}{Q''}$, etc., being the coefficients of the binomial formula.

Hence the binomial formula is true when the exponent of the required power is fractional.

3°. The given exponent is negative, and either entire or fractional.

In the equation $f(m).f(n)=f(m+n)$, let $n=-m$; then, $f(m).f(n)=f(m).f(-m)=f(m-m)=f(0)=1$.

$$\text{Hence,} \quad f(-m) = \frac{1}{f(m)}. \quad (\text{iv})$$

Now $f(m)=(1+x)^m$, whether m be integral or fractional;

$$\text{whence} \quad f(-m) = \frac{1}{(1+x)^m} = (1+x)^{-m} \dots \quad (\text{v})$$

that is, with the meaning that we have assigned to the functional symbol $f(\quad)$,

$$(1+x)^{-m} = 1 - mx + \frac{-m(-m-1)}{2} x^2 + \frac{-m(-m-1)(-m-2)}{1.2.3} x^3 + \dots$$

In this equation place $x = \frac{y}{z}$, and multiply both members by z^{-m} . There results,

$$(z+y)^{-m} = z^{-m} + (-m)z^{-m-1}y + \frac{-m(-m-1)}{2}z^{-m-2}y^2 + \dots$$

In other words, the binomial formula is true for all powers of which the exponents are negative. The truth of the theorem is, therefore, fully established.

202. Corollary I.—*The term of the binomial formula that has t terms before it, is called the General Term, and is*

$$\frac{m(m-1)(m-2)(m-3)\dots(m-t+1)}{1.2.3.4\dots t} x^{m-t} y^t.$$

For, it will be seen that the last factor in the denominator of each coefficient equals the number of preceding terms, and the last factor of the numerator is always m diminished by a quantity less by 1 than the last factor of the denominator; the exponent of x is always m diminished by the number of preceding terms, and the exponent of y is always the number of preceding terms. Hence, when t is the number of preceding terms we have the foregoing form. This is called the *general term*, for the reason that by substituting 1, 2, 3, etc., for t , any term of the formula, after the first, may be obtained.

Thus, to get the sixth term we place $t=5$, the number of terms before the sixth, and obtain

$$\frac{m(m-1)(m-2)(m-3)(m-4)}{1.2.3.4.5} x^{m-5} y^5.$$

Similarly, to get the tenth term we place $t=9$, whence $m-t+1=m-9+1=m-8$ and we obtain

$$\frac{m(m-1)(m-2)\dots(m-8)}{1.2.3.4.5.6.7.8.9} x^{m-9} y^9.$$

In the development of $x+y$, or any other in which the exponent of each term of the given binomial is 1, the sum of the exponents of these terms in any term of the formula, is equal to the exponent of the required power.

203. Number of Terms.

The number of terms in the second member of the binomial formula is $m+1$ when m is positive and integral, and is infinite when m is negative or fractional.

For, it takes m subtractions to reduce m to zero, if we diminish it by 1 at each subtraction. Now, we diminish the exponent m by 1 in the second term, by 2 in the third, and so on. Hence, when we have subtracted m from it, the number of terms in which the subtraction has taken place, *i. e.*, the number of terms after the first, will be m , and therefore the whole number of terms will be $m+1$.

If, however, m be negative, it is evident that by subtracting 1 for each successive term, we increase m numerically. Hence, as we go on subtracting, the result increases numerically (decreasing algebraically), and will therefore never be zero. Hence we shall never get to the last term, *i. e.*, the number of terms is infinite. If, again, m be fractional, the successive subtraction of 1 will at some point give a negative result; we shall therefore never reach the last term; *i. e.*, the number of terms is infinite.

204. Corollary II.—*The coefficients of any two terms of the development of $(x+y)^m$ equally distant from the extremes, (*i. e.*, the first and last terms respectively) are equal.*

$$(1.) (x+y)^m = x^m + mx^{m-1}y + m\frac{m-1}{2}x^{m-2}y^2 + \dots + y^m =$$

$$(y+x)^m = y^m + mxy^{m-1} + m\frac{m-1}{2}x^2y^{m-2} + \dots + x^m (2.)$$

As $x+y = y+x$, $(x+y)^m = (y+x)^m$, the second members of (1) and (2) must be the same, as each is the expression for the m th power of $x+y$. Hence the first term of (1) is the last term of (2), the second term of (1) the next to the last in (2), and so on. The term of (1) having r terms after it is the same as the term of (2) having r terms before it. We see, however, that the corresponding coefficients of (1) and (2) are equal; the coefficient of the second term of (1) equals

the coefficient of the second term of (2), and so on. Then the coefficient of that term of (1) that has r terms before it equals the coefficient of that term of (2) that has r terms before it. Hence the coefficient of that term of (2) that has r terms before it equals the coefficient of that term of (2) that has r terms after it, and so with (1). Hence the coefficients of those terms of (1) and (2) equally distant from the extremes are equal.

As x and y , in the binomial formula, may represent any quantities whatever, we deduce, for the involution of binomials, the following

RULE.—*Substitute the first term of the given binomial for x , the second term for y , and the exponent of the required power for m , in the formula:*

$$(x+y)^m = x^m + mx^{m-1}y + m\frac{m-1}{2}x^{m-2}y^2 + m\frac{m-1}{2} \cdot \frac{m-2}{3}x^{m-3}y^3 +$$

The second member of the formula will then be the required power of the given binomial.

EXAMPLES.

Expand the following by the binomial formula:

1. $(1+x)^{-3}$, $(1+x)^{\frac{5}{6}}$, $(1+x)^{\frac{1}{3}}$.

2. $(x+y)^{\frac{1}{2}}$, $(x-y)^6$, $(x-y)^{\frac{1}{4}}$.

3. $(x^2-y^2)^5$, $(x^2+y^2)^{\frac{1}{2}}$, $(x^3+y^3)^{-3}$, $(x^3+y^3)^{-\frac{5}{2}}$.

4. $(2x-1)^7$, $(2a^2-\frac{b}{2})^6$, $(a^2+bx^2)^{\frac{1}{2}}$.

Ans. $128x^7-448x^6+672x^5-560x^4+280x^3-84x^2+14x-1,$

$$32a^{10}-40a^8b+20a^6b^2-5a^4b^3+\frac{5}{8}a^2b^4-\frac{b^5}{32},$$

$$a+\frac{b}{2a}x^2-\frac{b^2}{8a^3}x^4+\frac{b^3}{16a^5}x^6-\dots\dots\dots$$

$$5. (a^{\frac{1}{2}} - b^{\frac{5}{2}})^{\frac{1}{2}}, (1-x)^{-\frac{1}{2}}, (a^2+x^2)^{-\frac{1}{2}}.$$

$$\text{Ans. } a^2 - 5a^{\frac{1}{2}}b^{\frac{5}{2}} + 10a^{\frac{3}{2}}b^{\frac{5}{2}} - 10a^{\frac{5}{2}}b^{\frac{5}{2}} + 5a^{\frac{3}{2}}b^{\frac{5}{2}} - b^{\frac{5}{2}},$$

$$1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \frac{35}{128}x^4 + \dots$$

$$a^{-\frac{1}{2}} - \frac{1}{4}a^{-\frac{5}{2}}x^2 + \frac{5}{32}a^{-\frac{3}{2}}x^4 - \frac{15}{128}a^{-\frac{1}{2}}x^6 + \dots$$

$$6. (a+y)^0, (a^2-b^2x^2)^{-\frac{5}{2}}, (x^2-4y^2)^{-\frac{3}{2}}.$$

$$7. (x^2+a^4)^{-\frac{1}{2}}, (m^2+x^2)^{-\frac{1}{2}}, (1+x^2)^{-\frac{1}{2}}.$$

$$\text{Ans. } x^{-1} - \frac{a^4}{2}x^{-3} + \frac{1.3}{2.4}a^8x^{-5} - \frac{1.3.5}{2.4.6}a^{12}x^{-7} + \frac{1.3.5.7}{2.4.6.8}a^{16}x^{-9} \dots$$

$$m^{-\frac{2}{3}} - \frac{1}{3}m^{-\frac{8}{3}}x^2 + \frac{1.4}{3.6}m^{-\frac{14}{3}}x^4 - \frac{1.4.7}{3.6.9}m^{-\frac{20}{3}}x^6 + \frac{1.4.7.10}{3.6.9.12}m^{-\frac{26}{3}}x^8 \dots$$

$$1 - \frac{x^2}{5} + \frac{6.x^4}{5.10} - \frac{6.11.x^6}{5.10.15} + \frac{6.11.16.x^8}{5.10.15.20} - \dots$$

$$8. (a^2+x^2)^{-n}, (x^2-z^2)^{-\frac{1}{n}}, (r^2-s^2)^{-\frac{5}{2}}, (r^2-s^2)^{\frac{3}{2}}.$$

205. Involution of Polynomials.

To Deduce a Rule for raising a polynomial to any power.

Since x and y in the binomial formula represent any quantities whatever, we may let x represent one term or the sum of several terms of any given polynomial, and y the sum of the remaining terms. The given polynomial will then be represented by $x+y$, and we may obtain its n th power by raising $x+y$ to the n th power by the binomial formula, and then substituting, in the result, for x and y their assumed values. As we have proved the binomial formula true for every value of n , we can raise a polynomial to any power of which the exponent is either integral or fractional by the method here indicated. In other words, since n represents any quantity, all powers of any polynomial are formed according to the same general law.

From the foregoing considerations we deduce the following

RULE—Represent the sum of several terms of the given polynomial by $\mathbf{x}$, and the sum of the remaining terms by $\mathbf{y}$. Expand $\mathbf{x} + \mathbf{y}$ to the required power by the binomial formula. Substitute in the result for $\mathbf{x}$ and $\mathbf{y}$ their assumed values.

206. Scholium.—When the exponent of the required power is a positive integer, very little is saved by the application of the rule; when this exponent is negative or fractional, however, the method here laid down—or the application of the resulting formulæ (see examples)—is the simplest.

EXAMPLES.

Expand the following expressions:

1. $(a+b+c)^2, (a+b+c)^3, (a+b+c+d+e+\dots)^2.$

Ans. $a^2+b^2+c^2+2ab+2ac+2bc$

$a^3+b^3+c^3+3a^2b+3a^2c+3b^2c+3ab^2+3ac^2+3bc^2+6abo$

$$\left\{ \begin{array}{l} a^2+b^2+c^2+d^2+e^2+\dots \\ +2ab+2ac+2ad+2ae+\dots \\ +2bc+2bd+ \\ +2cd+2ce+\dots \\ +\dots\dots\dots \end{array} \right\} = \left\{ \begin{array}{l} a^2+b^2+c^2+d^2+\dots \\ +2a(b+c+d+\dots) \\ +2b(c+d+e+\dots) \\ +2c(d+e+f+\dots) \\ +\dots\dots\dots \end{array} \right.$$

2. $(a+b+c+d+e+\dots)^3, \left(2-x^{\frac{1}{2}}+\frac{x}{2}\right)^2, (1-x^{\frac{1}{2}}+x)^2.$

Ans.

$$\left\{ \begin{array}{l} a^3+b^3+c^3+d^3+e^3+\dots \\ +3a^2b+3a^2c+3a^2d+\dots \\ +3b^2c+3b^2d+3b^2e+\dots \\ +3c^2d+3c^2e+\dots \\ +\text{etc.} \\ +3ab^2+3ac^2+3ad^2+\dots \\ +3bc^2+3bd^2+3be^2+\dots \\ +3cd^2+3ce^2+\dots \\ +\text{etc.} \\ +6abc+6abd+\dots+6acd+\dots \\ +6bcd+6bce+\dots+6bde+\dots \\ +6cde+6cdf+\dots+6cef+\dots \end{array} \right.$$

$4-4x^{\frac{1}{2}}+3x-x^{\frac{3}{2}}+\frac{x^2}{4}; 1-2x^{\frac{1}{2}}+3x-2x^{\frac{3}{2}}+x^2$

$$3. \left(2x^{-\frac{1}{2}} + \frac{1}{2} + 2x^{\frac{1}{2}}\right)^2, (3x^n - 2x^{n+1} + x^{n+2})^2, (2x^n - 1 + 2x^n)^2.$$

$$\text{Ans. } 4x^{-1} + 2x^{-\frac{1}{2}} + \frac{33}{4} + 2x^{\frac{1}{2}} + 4x$$

$$\begin{aligned} & 9x^{2n} - 12x^{2n+1} + 10x^{2n+2} - 4x^{2n+3} + x^{2n+4} \\ & 8x^{3n} - 12x^{2n} + 30x^n - 25 + 30x^n - 12x^{2n} + 8x^{3n} \end{aligned}$$

$$4. (1+5x+7x^2+9x^3+\dots)^2, (3+7x+11x^2+14x^3)^4$$

$$(5x^2-2x+1+3x^{-1}+x^{-2})^3$$

$$5. \left(1 + \frac{1}{3}x + \frac{1}{5}x^2 + \frac{1}{7}x^3\right)^{\frac{1}{2}}, (a+bx+cx^2+dx^3)^{\frac{1}{3}}$$

6. The foregoing examples furnish general formulæ for involving polynomials. Give, in common language, the expression for the square of a polynomial; for the cube of a polynomial.

$$7. (a+b+c+d+e+\dots)^2 \text{ may be written}$$

$$\begin{cases} a^2 \\ +2ab+b^2 \\ +2(a+b)c+c^2 \\ +2(a+b+c)d+d^2 \\ +2(a+b+c+d)e+e^2 \\ +\text{etc., etc.} \end{cases}$$

Again:

$$(a+b+c+d+e+\dots)^2 =$$

$$\begin{cases} a^2 \\ +3a^2b+3ab^2+b^3 \\ +3(a+b)^2c+3(a+b)c^2+c^3 \\ +3(a+b+c)^2d+3(a+b+c)d^2+d^3 \\ +3(a+b+c+d)^2e+3(a+b+c+d)e^2+e^3 \\ +\text{etc., etc.} \end{cases}$$

$$(a+b+c+d+e+\dots)^4 =$$

$$\begin{cases} a^4 \\ +4a^3b+6a^2b^2+4ab^3+b^4 \\ +4(a+b)^3c+6(a+b)^2c^2+4(a+b)c^3+c^4 \\ +4(a+b+c)^3d+6(a+b+c)^2d^2+4(a+b+c)d^3+d^4 \\ +\text{etc., etc.} \end{cases}$$

What is the expression for $(a+b+c+d+e+\dots)^n$ for all values of n ?

SECTION II.

EVOLUTION, OR EXTRACTION OF
ROOTS.

207. Evolution is the process of extracting any root of a given quantity, *i. e.*, of dividing the given quantity into any number of equal factors. It is the converse of Involution.

208. The Degree of the root is indicated by the number of equal factors, which gives also, the *name* to the root. Thus 7 is the square root or second root of 49, the cube root or third root of 343, and the fourth root of 2401. *Square* roots are of the *second* degree, *cube* roots of the *third*, *fourth* roots of the *fourth* degree, *nth* roots of the *nth* degree, and so on.

209. A Perfect Power of any degree is one that can be divided into the number of equal factors required.

A literal quantity is said to be a perfect power of any degree when the exponent of each letter is exactly divisible by the number indicating the degree.

210. An Imperfect Power of any degree is a quantity that cannot be divided into the required number of equal factors.

211. The same quantity may be a perfect power of one degree and an imperfect power of another. Thus 27 is an *imperfect* power of the *second* degree, but a *perfect* power of the *third*. It cannot be divided into *two* equal factors, but may be divided into *three*. 64 is a perfect power of the third degree (4 being the root), and also of the sixth degree (2 being the root.)

212. The sign of Evolution is the character $\sqrt{}$. The degree of the root required is indicated by a symbol placed within the opening of the sign. This symbol is called the **Index**. If no Index be expressed, 2 is understood.

EVOLUTION OF MONOMIALS.

213. To Deduce a Rule for extracting any root of a Monomial.

I. Law of Coefficients and Exponents.

Let it be required to extract the m th root of $3^{2m}a^mb^6mc^5m$. This means that we are required to find a quantity which, raised to the power m , will produce the given quantity. From (195) we know that the coefficient of the root must be such that, raised to the power m , it will produce 3^{2m} . We also know that each letter in the root will be found in the power, and that the exponent of any letter in the m th power of the root (*i. e.*, in the given quantity) will equal the exponent of the same letter in the root, multiplied by m . Hence we see that the coefficient of the root must be 3^2 or 9; that the exponent of a in the root must be 1, of b , 6, and of c , 5. Hence the required root is $9ab^6c^5$. Reviewing our work we see that the root may be obtained at once by extracting the required root of the coefficient for the new coefficient, and after this writing the letters found in the given quantity, giving to each letter an exponent equal to its original exponent divided by the index of the root.

II. Law of Signs.

From (198) we see that *even* powers of all quantities are *positive*, whether the original quantities be positive or negative. Hence, if the index of the required root be even, and the given quantity positive, the root may be either positive or negative, and we prefix the double sign. If the index be even and the given quantity negative, the root cannot be extracted, for no quantity raised to an even power will give a negative result.

Finally, if the index be odd, the root will (198) be positive or negative, according as the given quantity is positive or negative.

Our reasoning being perfectly general, we deduce from the foregoing considerations the

RULE.—*Extract the required root of the coefficient for a new coefficient. After this, write all the letters entering the given quantity, giving to each an exponent equal to its exponent in the given quantity, divided by the index of the root.*

If the index be odd, prefix to the root the sign of the given quantity. If the index be even and the given quantity positive, prefix to the root the double sign. If the index be even and the quantity negative, the root cannot be extracted.

214. *The n th root of an entire imperfect n th power cannot be expressed in exact parts of 1.*

Let x represent any entire or integral imperfect n th power. Then $\sqrt[n]{x}$ cannot be exactly expressed in parts of 1. It cannot be expressed in entire quantities, for if it could be so expressed, x would be a perfect n th power. Hence if $\sqrt[n]{x}$ can be exactly expressed it must be in the form of a fraction. Suppose, then, $\sqrt[n]{x} = \frac{a}{b}$, the fraction $\frac{a}{b}$ being in its simplest form. Since $\sqrt[n]{x}$ and $\frac{a}{b}$ are equal, the results of taking them the same number of times as factors must be equal. Hence $\sqrt[n]{x}$ taken n times as a factor equals $\frac{a^n}{b^n}$. Since $\sqrt[n]{x}$ is one of the n equal factors into which x is divided, if we take the quantity n times as a factor we must obtain x . Hence $x = \frac{a^n}{b^n}$. Since a and b have no common factors a^n and b^n have no common factors, i. e., $\frac{a^n}{b^n}$ is an irreducible fraction. Hence we have an entire quantity equal to an irreducible fraction, which is absurd. Hence $\sqrt[n]{x}$ cannot be

equal to $\frac{a}{b}$. Since $\sqrt[n]{x}$ cannot be expressed exactly either in entire quantities or in a fractional form, it cannot be expressed exactly in parts of 1.

215. Corollary I.—Any root of a Fraction may be extracted by the following

RULE.—Extract the indicated root of the numerator, for a new numerator, and the indicated root of the denominator, for a new denominator.

This follows directly from (213); for, let $\sqrt[n]{\frac{a}{b}}$ be required;

$$\frac{a}{b} = ab^{-1}. \therefore \sqrt[n]{\frac{a}{b}} = \sqrt[n]{ab^{-1}} = a^{\frac{1}{n}} b^{-\frac{1}{n}} \text{ (213),}$$

$$\text{and } a^{\frac{1}{n}} b^{-\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Or, we may deduce this rule from (195), thus:—Since from definition $\sqrt[n]{\frac{a}{b}}$ must be a fraction, which raised to the n th power, will produce $\frac{a}{b}$, the n th power of the numerator of this root-fraction must (195) be a , and the n th power of the denominator must be b . From the definition of power and root, a quantity, which raised to the n th power, produces a given quantity, must be the n th root thereof; hence the required numerator must be $\sqrt[n]{a}$, and the required denominator $\sqrt[n]{b}$, so that $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

216. Corollary II.—Since a MIXED QUANTITY may always be transformed into a fraction we have, for the **Evolution of Mixed Quantities**, the following

RULE.—Transform the given quantity into an equivalent

fraction ; extract the required root of this fraction, and transform the result into a mixed quantity.

217. From (213) it follows that

1°. *The nth root of the product of several quantities equals the product of the nth roots of the quantities.*

And from (215) that

2°. *The nth root of any quotient equals the nth root of the dividend, divided by the nth root of the divisor.*

EXAMPLES.

1. $\sqrt[3]{36a^2b^3c^{2m}} = ? \quad \sqrt[5]{32a^{10}(b+c)^{15}x^5y^{20}} = ?$
 $\sqrt[3]{1728a^{36}b^{12}(c+d)^{6x}y^3} = ? \quad \sqrt[7]{128x^{-14}(y+z)^{-7}a^{14}} = ?$
2. $\sqrt[n]{a^{6n}(x+y)^{n^2+2n}a^{75n}z^n} = ? \quad \sqrt[6]{4096x^{12}(c+d)^6y^{18}} = ?$
3. $\sqrt[3]{\frac{3375x^3y^{12}z^6}{4913(c+d)^{13}}} = ? \quad \sqrt[9]{-512a^{16}b^{-86}(x+y)^{-\frac{54}{100}}} = ?$
4. $\sqrt{-1131a^6(x+y)^{\frac{21}{10}}(c+d)^{-\frac{1}{3}}} = ? \quad \sqrt[6]{729(x+y)^{-\frac{1}{2}}d^{12}c^{18}} = ?$

EVOLUTION OF POLYNOMIALS.

218. To Deduce a 'Rule for the extraction of any root of a polynomial.

I. Extraction of the square root of polynomials.

Let P be any polynomial, arranged according to the descending powers of some letter. Since the square of a polynomial is obtained by multiplying the polynomial by itself, the first term of this square arranged according to the descending powers of any letter is (67) obtained *without reduction*, by squaring the first term of the polynomial, arranged according to the descending powers of the same

letter. Hence, P BEING ARRANGED AS STATED, THE SQUARE ROOT OF ITS FIRST TERM WILL BE THE FIRST TERM OF THE SQUARE ROOT. Represent this term, whatever it may be, by r , and the following terms of the root, arranged according to the descending powers of the leading letter of P, by r' , r'' , r''' , and so on. Let $s=r'+r''+r''' + \dots$. Then

$$\begin{aligned} P &= (r+r'+r''+r''' + \dots)^2 = (r+s)^2 \\ &= r^2 + 2rs + s^2 \\ &= r^2 + 2r(r'+r'' + \dots) + (r'+r'' + \dots)^2 \end{aligned}$$

Having obtained r , let us subtract its square from P, so as to simplify our search for the remaining terms of the root. We thus obtain

$$\begin{aligned} P - r^2 &= R_1 = 2r(r'+r''+r''' + \dots) + (r'+r''+r''' + \dots)^2 \\ &= 2rr' + 2rr'' + 2rr''' + \dots \end{aligned}$$

Since r' has a higher power of the leading letter than any following term in the root, and r has a higher power of this letter than *any* other term of the root, $2rr'$ will contain a higher power of the leading letter than any other term of the product $2r(r'+r'' + \dots)$. For a reason already given r'^2 will contain a higher power of the leading letter than any other term of $(r'+r''+r''' + \dots)^2$. But since r contains a higher power of the leading letter than r' , $2rr'$ contains a higher power of this letter than $r'^2 = r'r'$. The term $2rr'$ then contains a higher power of the leading letter than *any other term* of the remainder R_1 , and will therefore not reduce with any other term. Hence, $2rr'$ appears, *without reduction*, as the first term of our first remainder, which is arranged as was the polynomial P. By inspection, we see that if $2rr'$ be divided by $2r$, the quotient, r' , will be the next term of the root. Hence

IF THE FIRST TERM OF THE FIRST REMAINDER BE DIVIDED BY TWICE THE FIRST TERM OF THE ROOT, THE QUOTIENT WILL BE THE SECOND TERM OF THE ROOT.

Having now obtained $r+r'$, let us subtract the square of this part of the root from P, so as to simplify our search for the remaining terms of the root.

To do this, place $r+r'=t$ and $r''+r''' + \dots = s'$. Then $P=(t+s')^2=t^2+2ts'+s'^2$. Subtracting from this t^2 , we have

$$\begin{aligned} P-t^2=R_2 &= 2ts' + s'^2. \\ &= 2(r+r')(r''+r''' + \dots) + (r''+r''' + \dots)^2. \\ &= 2rr'' + 2rr''' + \dots \end{aligned}$$

By reasoning on this remainder as upon the first, we may show that $2rr''$ will contain a higher power of the leading letter than *any other term* of R_2 . It will therefore not reduce with any other term, and will appear *without reduction* as the first term of the arranged remainder R_2 . By inspection we see that $2rr''$ divided by $2r$ gives as a quotient r'' , the next term of the root. Hence

IF THE FIRST TERM OF THE SECOND REMAINDER BE DIVIDED BY TWICE THE FIRST TERM OF THE ROOT THE QUOTIENT WILL BE THE THIRD TERM OF THE ROOT.

Placing $r+r'+r''=t'$ and $r''' + r'''' + \dots = s''$,

$$P=(t'+s'')^2=t'^2+2t's''+s''^2.$$

Subtracting t'^2 for a reason already given,

$$\begin{aligned} P-t'^2=R_3 &= 2t's'' + s''^2 = 2(r+r'+r'')(r''' + \dots) + s''^2. \\ &= 2rr''' + 2rr'''' + \dots \end{aligned}$$

Reasoning as before, we may show that $2rr'''$ appears, *without reduction*, as the first term of the arranged remainder R_3 , and therefore that

IF THE FIRST TERM OF THE THIRD REMAINDER BE DIVIDED BY TWICE THE FIRST TERM OF THE ROOT, THE QUOTIENT WILL BE THE FOURTH TERM OF THE ROOT.

From the nature of the reasoning we shall see, on continuing this method of operating, that

IF THE FIRST TERM OF ANY REMAINDER BE DIVIDED BY TWICE THE FIRST TERM OF THE ROOT, THE QUOTIENT WILL BE A NEW TERM OF THE ROOT.

Since $t=r+r'$, $t^2=r^2+2rr'+r'^2=r^2+(2r+r')r'$, and $P-t^2=R_2=P-r^2-2(r+r')r'$; $P-r^2=R_1$. Hence $P-t^2=R_2=R_1-(2r+r')r'$. Hence the second remainder, instead of being obtained by subtracting $t^2=(r+r')^2$ from P , may be obtained by subtracting $(2r+r')r'$ from the first remainder.

Similarly, the third remainder may be found by subtracting $(2r+r'')r''$ from the second remainder, and so on for all the remainders.

If (67) we had arranged P according to the ascending powers of some letter, a similar course of reasoning would evidently have led to the same conclusion.

From the foregoing considerations we deduce the

RULE.—*Arrange the given polynomial according to the ascending or descending powers of some letter. Extract the square root of the first term for the first term of the root. Subtract the square of the first term of the root from the given polynomial. Divide the first term of this remainder by twice the first term of the root. Add the quotient to the root and to the divisor. Multiply the augmented divisor by the second term of the root and subtract the product from the first remainder.*

Divide the first term of the second remainder by twice the first term of the root, for the third term of the root. Multiply twice the root before found, plus the third term, by the third term, and subtract the result from the second remainder.

Proceed in a similar manner until a remainder of zero is obtained, or until the first term of some remainder is not exactly divisible by the divisor. In the first case the polynomial is a perfect square; in the second it is an imperfect square, and its square root cannot be obtained exactly.

219. Scholium.—This rule may be applied to numbers by considering r , r' , etc., as indicating different orders of units. The principal difficulty is in finding the terms r^2 , etc., which will no longer be separate and distinct. We find them by pointing off periods of figures from the right. Thus if the number contain six places, it is between 1000000 and 10000. Its root then is between 1000 and 100, i. e., must be expressed by three figures. Let r represent the hundreds, r' the tens, and r'' the units of the root. r^2 cannot be less than 100. Hence the two figures on the right have nothing to do with r^2 , and we point them off. r^2 cannot be less than 10000, whence the last four figures on the right have nothing to do with r^2 , and we point off two more, making four, from the right of the number. r^2 must be in the two figures on the left of the number, and this is the best that we can do towards sepa-

given polynomial, and dividing the first term of the remainder by $2(8x^2)$ we find, as the quotient, and therefore the second term of the root, $12x^2y$. Multiplying $2(8x^2)+12x^2y$ by $12x^2y$ and subtracting the product from the first remainder, we obtain, as the first term of the second remainder, $96x^4y^2$, which, divided by $2(8x^2)$, gives $6xy^2$ as the third term of the root. Multiplying $2(8x^2+12x^2y)+6xy^2$ by $6xy^2$ and subtracting the product from the second remainder, we obtain $16x^2y^3$ as the first term of the third remainder. Dividing $16x^2y^3$ by $2(8x^2)$, we obtain y^3 as the fourth term of the root, and multiplying and subtracting as before, we obtain a remainder of zero. Hence $8x^2+12x^2y+6xy^2+y^3$ is the required square root; if squared, it will give the original polynomial. $-8x^2-12x^2y-6xy^2-y^3$, if squared, will also give the original polynomial; hence we have as the square root, $\pm(8x^2+12x^2y+6xy^2+y^3)$. As the double sign should really precede all roots of an even degree, it will hereafter be understood before such roots, no sign being written.

$$7. \sqrt{\left(4b^2-12ab+11a^2-\frac{3a^3}{b}+\frac{a^4}{4b^3}\right)}=? \text{ Ans. } 2b-3a+\frac{a^2}{2b}.$$

$$8. \sqrt{(1+4x+2x^2+9x^4-4x^5+4x^6)}=? \text{ Ans. } 1+2x-x^2+2x^3.$$

$$9. \sqrt{(1+4x+10x^2+12x^3+9x^4)}=? \text{ Ans. } 1+2x+3x^2.$$

$$10. \sqrt{(a^2+2abx+(b^2+2ac+c^2x^2)x^2+2(bc+ad)x^3+2bdx^4+2cdx^5+d^2x^6)}=? \text{ Ans. } a+bx+cx^2+dx^3.$$

$$11. \sqrt{\left(\frac{4}{a^2}-\frac{4}{a^3b^2}+\frac{13}{a^4b^4}-\frac{6}{a^5b^6}+\frac{9}{a^6b^8}\right)}=? \text{ Ans. } \frac{2}{a}-\frac{1}{a^2b^2}+\frac{3}{a^3b^4}.$$

$$12. \sqrt{\left(\frac{4a^2}{9b^4}-\frac{4ac}{3b^2}+5c^2+27\frac{b^4c^5}{a^3}+81\frac{b^8c^8}{4a^4}\right)}=? \text{ Ans. } \frac{2}{3}\frac{a}{b^2}-c+\frac{3b^3c^3}{a}+\frac{9b^4c^2}{2a^2}.$$

$$13. \sqrt{\left(2+3x^{\frac{1}{2}}+x+1+3x^{\frac{1}{2}}+2x+2+3x^{\frac{1}{2}}+2x\right)}=?$$

$$\text{Ans. } 3+5x^{\frac{1}{2}}+3x.$$

$$14. \sqrt{1-x}=? \quad \text{Ans. } 1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^4-\dots$$

$$15. \sqrt{4x^4-12x^5+2x^7+9x^8-4x^{11}+16x^{13}-19x^{16}+24x^{19}+\frac{x^{10}}{4}}=?$$

$$\sqrt{x^4+y^4+z^4+w^4-2x^2(y^2+w^2)-2y^2(z^2-w^2)+2z^2(x^2-w^2)}=?$$

221. Proposition I.—*A binomial is never a perfect square.*

For, its root cannot be a monomial, since the square of a monomial is a monomial. The root cannot be a polynomial, for the square of the simplest polynomial (a binomial) has three terms.

Proposition II.—*A trinomial is a perfect square when, the trinomial being arranged according to the powers of some letter, the square of the middle term equals four times the product of the extremes.*

See (112). The square root of such a trinomial is the sum or difference of the square roots of the extreme terms.

Proposition III.—*A quadrinomial is never a perfect square, for, its square root cannot be a monomial or a binomial, for reasons indicated above. It cannot be a trinomial, for the square of a trinomial has at least five terms, and the square of any other polynomial must have a greater number of terms.*

222. II.—**To Deduce the Rule for extracting the cube root of a polynomial.**

Let P be any polynomial, arranged according to the descending powers of some letter. The cube root of P is required. By (192) P is the product obtained by taking the cube root three times as a factor. Hence the first term

of P (as arranged), is an irreducible term of the product, and was obtained, *without reduction*, by multiplying the first term of the cube root (arranged as P has been) by itself twice, i. e., cubing it. Hence

THE CUBE ROOT OF THE FIRST TERM OF P IS THE FIRST TERM OF THE CUBE ROOT.

Let r represent this term, and r' , r'' , r''' , etc., the succeeding terms of the root, whatever they may be, arranged according to the descending powers of the leading letter of P . Let $r' + r'' + r''' + \dots = s$. Then

$$P = (r + s)^3 = r^3 + 3r^2s + 3rs^2 + s^3.$$

Having obtained r , we subtract r^3 from P , to simplify our search for the remaining terms of the root. Denoting the remainder by R_1 , we have,

$$\begin{aligned} P - r^3 &= R_1 = 3r^2s + 3rs^2 + s^3. \\ &= 3r^2(r' + r'' + \dots) + 3r(r' + r'' + \dots)^2 + (r' + r'' + \dots)^3 \\ &= 3r^2r' + 3r^2r'' + \dots \end{aligned}$$

(substituting for s its value, $r' + r'' + r''' + \dots$)

Since r contains a higher power of the leading letter than any other term of the root, and r' a higher power than any other term of s , the product of $3r^2$ and r' will contain a higher power of the leading letter than any other term of the product $3r^2(r' + r'' + r''' + \dots)$, and will not reduce with any such term. For the same reason $3rr'^2$ will contain a higher power of the leading letter than any other term of the product $3r(r' + r'' + r''' + \dots)^2$, and will not reduce with any such term. Again, since r' contains a higher power of the leading letter than any other term of s , $= r' + r'' + \dots$, r'^3 will contain the highest power of this letter in $(r' + r'' + r''' + \dots)^3$, and will not reduce with any other term of s^3 .

Now $3rr'^2 = 3rr'r'$ contains a higher power of the leading letter than $r'^3 = r'.r'.r'$, since r contains a higher power of this letter than r' ; the factor $r'.r' = r'^2$ is common to these two terms, and there remains the factor $3r$ of the first, containing a higher power of the leading letter than the factor r' of the second, whence $3rr'^2$ contains a higher power of the leading letter than r'^3 . Finally $3r^2r' = 3rr'r'$,

contains a higher power of the leading letter than $3rr'^2 = 3r'r'r'$, since r contains a higher power of this letter than r' .

$3r'^2r'$, then, contains a higher power of the leading letter than any other term of the remainder R_1 , and will therefore not reduce with any other term. Hence $3r'^2r'$ appears, *without reduction*, as the first term of our first remainder, arranged according to the descending powers of the leading letter of P . An inspection of this term shows that, if it be divided by $3r'^2$, the quotient, r' , will be the next term of the root. Hence,

IF THE FIRST TERM OF THE FIRST REMAINDER BE DIVIDED BY THREE TIMES THE SQUARE OF THE FIRST TERM OF THE ROOT THE QUOTIENT WILL BE THE SECOND TERM OF THE ROOT.

Let us now subtract $(r+r')^2$ from P , for a reason already indicated. To do this, place $r+r'=t$; $r''+r'''+\dots=s'$. Then the root will be indicated by $t+s'$, and

$$P=(t+s')^2=t^2+3t's'+3t's'^2+s'^2,$$

from which, subtracting t^2 , $R_1=P-t^2=3t's'+3t's'^2+s'^2$, or, substituting the values of t and s' ,

$$\begin{aligned} R_1 &= P - t^2 = 3(r+r')(r''+r'''+\dots) + 3(r+r')(r''+r'''+\dots)^2 \\ &\quad + (r''+r'''+\dots)^2, \\ &= 3r^2r'' + 3r^2r''' + \dots \end{aligned}$$

Reasoning upon this remainder as upon the preceding, we can show that $3r^2r''$ will contain a higher power of the leading letter than any other term of the second remainder, and will therefore appear, *without reduction*, as the first term of the arranged second remainder. If $3r^2r''$ be divided by $3r^2$ the quotient will be r'' , the third term of the root. Hence,

IF THE FIRST TERM OF THE SECOND REMAINDER BE DIVIDED BY THREE TIMES THE SQUARE OF THE FIRST TERM OF THE ROOT, THE QUOTIENT WILL BE THE THIRD TERM OF THE ROOT.

Placing $r+r'+r''=t'$, and $r''' + r^{iv} + \dots = u$,

$P=(t'+u)^2=t'^2+3t'u+3t'u^2+u^2$; subtracting t'^2 , for a reason already given,

$$\begin{aligned} R_2 &= P - t'^2 = 3t'u + 3t'u^2 + u^2, \\ &= 3(r+r'+r'')(r''' + r^{iv} + \dots) + 3(r+r'+r'')(r''' + r^{iv} + \dots)^2 \\ &\quad + (r''' + r^{iv} + \dots)^2. \end{aligned}$$

Upon this remainder we may reason as before, and show that $3r^2r'''$ appears, *without reduction*, as the first term of the arranged third remainder. $3r^2r''' \div 3r^2 = r'''$, the fourth term of the root. Hence,

IF THE FIRST TERM OF THE THIRD REMAINDER BE DIVIDED BY THREE TIMES THE SQUARE OF THE FIRST TERM OF THE ROOT, THE QUOTIENT WILL BE THE FOURTH TERM OF THE ROOT.

From (201-5) it follows that we could go on in a similar way, and show that

IF THE FIRST TERM OF ANY REMAINDER BE DIVIDED BY THREE TIMES THE SQUARE OF THE FIRST TERM OF THE ROOT, THE QUOTIENT WILL BE THE NEW TERM OF THE ROOT.

If (67) P had been arranged according to the ascending powers of any letter, we might evidently, reasoning in a similar manner, have arrived at the same conclusion.

From the foregoing considerations we deduce the

RULE.—*Arrange the given polynomial according to the ascending or descending powers of some letter. The cube root of the first term is the first term of the cube root.*

Subtract the cube of the first term of the root from the given polynomial. Divide the first term of the remainder by three times the square of the first term of the root. The quotient will be the second term of the root.

Cube the sum of the terms of the root now found, and subtract this cube from the given polynomial. Divide the first term of the remainder thus obtained by three times the square of the first term of the root. The quotient will be the third term of the root.

Subtract from the given polynomial the cube of that part of the root now found. Divide the first term of the resulting remainder by three times the square of the first term of the root. The quotient will be the fourth term of the root. Continue this process until a remainder of zero is obtained, or (if from the nature of the case, no such remainder will ever result), as far as desired.

223. Scholium I.—If a remainder of zero results, the given polynomial is a perfect cube. If a remainder of zero cannot be obtained the polynomial is an imperfect cube.

224. Scholium II.—The preceding discussion may be applied to numbers by letting, r , r' , r'' , etc. represent the figures indicating the different orders of units. Thus, if r represent hundreds, r' represents tens and r'' units. Then P represents the cube of a number, consisting of hundreds, tens, and units. The terms r^3 , $3r^2r'$ etc., will not, in the case of numbers, be distinct, as in the case just discussed. We must, therefore, pick out the cube of the hundreds, of the tens, and so on, in some special way. As the cube of 100 is 1000000, the cube of the number of hundreds in the required root cannot be less than 1000000. Hence the six figures on the right of the number can have nothing to do with this cube, and we point them off, to separate them from the figures in which the cube of the hundreds must occur. This is the best that we can do toward separating the cube of the hundreds from the other parts of the number, and we take the cube root of the greatest perfect cube in this left hand part, as the number of hundreds in the root. Subtracting this perfect cube from the number, we must seek the tens. The cube of the tens, the lowest number we have to deal with in the second step, cannot be less than 1000. Hence the three figures on the right of the number can have nothing to do with it, and we point them off. As $3r^2r'$ cannot be less than 100000, the two right hand figures of the middle period can have nothing to do with it, and we leave them out of consideration. Leaving out these, and those previously pointed off, then, we proceed to divide the rest of the remainder by $3r^2$, to get r' , the number of tens, having obtained which, we cube the root found, and so continue in a manner analogous to that laid down in the rule, except that the trial divisor, instead of being always $3r^2$, is three times the square of the part of the root already found.

$$\begin{aligned} \text{225. Scholium III.} \text{—Since } R_2 &= P - r^3 = P - (r^3 + 3r^2r' + 3rr'^2 + r'^3) \\ &= P - r^3 - 3r^2r' - 3rr'^2 - r'^3 \\ &= P - r^3 - (3r^2r' + 3rr'^2 + r'^3) \end{aligned}$$

And since $R_1 = P - r^3$,

$$\begin{aligned} R_2 &= R_1 - (3r^2r' + 3rr'^2 + r'^3) \\ &= R_1 - (3(r^2 + rr')r' + r'^3) \\ &= R_1 - [(3r^2 + 3rr') + r'^3]r'. \end{aligned}$$

Hence the second remainder might have been found by adding to the divisor three times the product of the two terms of the root found, plus the square of the second term, multiplying this sum by the second term, and subtracting the result from the first remainder; similarly with the other remainders.

EXAMPLES.

$$\begin{array}{l}
 \text{1. Extract the cube root of } 8a^3 - 4a^2b + \frac{40b^4}{9a} - \frac{25b^5}{36a^2} - \frac{91}{27}b^3 \\
 + \frac{32}{3}ab^2 + \frac{125b^6}{216a^3} \\
 8a^3 - 4a^2b + \frac{32}{3}ab^2 - \frac{91}{27}b^3 + \frac{40b^4}{9a} - \frac{25b^5}{36a^2} + \frac{125b^6}{216a^3} \left| 2a - \frac{1}{3}b + \frac{5b^2}{6a} \right. \\
 8a^3
 \end{array}$$

$$\begin{array}{r}
 - 4a^2b + \frac{32}{3}ab^2 + \dots\dots\dots \\
 \hline
 \end{array} \quad (1)$$

$$\begin{array}{r}
 8a^3 - 4a^2b + \frac{2}{3}ab^2 - \frac{b^3}{27} \dots\dots\dots \left(2a - \frac{1}{3}b \right)^3 \\
 \hline
 \end{array}$$

$$\begin{array}{r}
 \frac{30}{3}ab^2 - \frac{90}{27}b^3 + \dots\dots\dots \\
 \hline
 \end{array} \quad (2)$$

$$\begin{array}{r}
 8a^3 - 4a^2b + \frac{32}{3}ab^2 - \frac{91}{27}b^3 + \frac{40b^4}{9a} - \frac{25b^5}{36a^2} + \frac{125b^6}{216a^3} \cdot \left(2a - \frac{b}{3} + \frac{5b^2}{6a} \right)^3 \\
 \hline
 0 \qquad \qquad \qquad (3)
 \end{array}$$

Arranging, with reference to the powers of a , and extracting the cube root of the first term, we obtain $2a$ as the first term of the required root. Cubing this, and subtracting the result from the given polynomial, we obtain $-4a^2b$ as the first term of the first remainder (1). Dividing this by $3(2a)^2 = 12a^2$, we obtain $\frac{b}{3}$ as the second term of the root. Subtracting $\left(2a - \frac{b}{3} \right)^3$ from the original polynomial we obtain $10ab^2$ as the first term of the second remainder (2), $10ab^2 \div 12a^2 = \frac{5}{6} \frac{b^2}{a}$, which is therefore the third term of the root. Subtracting from the given polynomial the cube of the root now found, we obtain a remainder of 0. Hence $2a - \frac{b}{3} + \frac{5}{6} \frac{b^2}{a}$ is the required root.

The given polynomial is a mixed quantity, and the root might also have been obtained, as indicated in (216), but

the general rule (222) is evidently the simpler here. The student must determine for himself which of the methods is better in any particular case; the rule (222) will, however, be found the more useful.

2.

$$\sqrt[3]{1+6x+9x^2+2x^3+15x^4+18x^5+30x^6-13x^7-12x^8+8x^9} = ?$$

$$\text{Ans. } 1+2x-x^2+2x^3.$$

3. Extract the cube root of

$$\frac{8y}{27x^{\frac{3}{2}}} + \frac{4y^{\frac{2}{3}}}{3x} + \frac{2y^{\frac{1}{3}}}{3x^{\frac{1}{2}}} - \frac{17}{3} - \frac{9x^{\frac{1}{2}}}{y^{\frac{1}{3}}} + \frac{5x}{y^{\frac{2}{3}}} + \frac{19x^{\frac{5}{2}}}{y} + \frac{6x^2}{y^{\frac{4}{3}}} - \frac{12x^{\frac{5}{2}}}{y^{\frac{5}{3}}} - \frac{8x^3}{y^2}$$

$$\frac{8y}{27x^{\frac{3}{2}}} + \frac{4y^{\frac{2}{3}}}{3x} + \frac{2y^{\frac{1}{3}}}{3x^{\frac{1}{2}}} + \dots \quad (1)$$

$$\frac{8y}{27x^{\frac{3}{2}}} + \frac{4y^{\frac{2}{3}}}{3x} + \frac{2y^{\frac{1}{3}}}{x^{\frac{1}{2}}} + 1 = \left(\frac{2y^{\frac{1}{3}}}{3x^{\frac{1}{2}}} + 1 \right)^3$$

$$- \frac{4y^{\frac{1}{3}}}{3x^{\frac{1}{2}}} - \frac{20}{3} \dots \quad (2)$$

$$\frac{8y}{27x^{\frac{3}{2}}} + \frac{4y^{\frac{2}{3}}}{3x} + \frac{2y^{\frac{1}{3}}}{3x^{\frac{1}{2}}} - 3 - \frac{3x^{\frac{1}{2}}}{y^{\frac{1}{3}}} + \frac{3x}{y^{\frac{2}{3}}} - \frac{x^{\frac{3}{2}}}{y} = \left(\frac{2y^{\frac{1}{3}}}{3x^{\frac{1}{2}}} + 1 - \frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} \right)^3$$

$$- \frac{8}{3} \dots \quad (3)$$

$$\frac{8y}{27x^{\frac{3}{2}}} + \frac{4y^{\frac{2}{3}}}{3x} + \frac{2y^{\frac{1}{3}}}{3x^{\frac{1}{2}}} - \frac{17}{3} - \frac{9x^{\frac{1}{2}}}{y^{\frac{1}{3}}} + \frac{5x}{y^{\frac{2}{3}}} + \frac{19x^{\frac{5}{2}}}{y} + \frac{6x^2}{y^{\frac{4}{3}}} - \frac{12x^{\frac{5}{2}}}{y^{\frac{5}{3}}} - \frac{8x^3}{y^2}$$

$$= \left(\frac{2y^{\frac{1}{3}}}{3x^{\frac{1}{2}}} + 1 - \frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} - \frac{2x}{y^{\frac{2}{3}}} \right)^3$$

0.

(4)

$$\text{Ans. } \frac{2y^{\frac{1}{3}}}{3x^{\frac{1}{2}}} + 1 - \frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} - \frac{2x}{y^{\frac{2}{3}}}.$$

$$4. \sqrt[3]{(27x-8x^{\frac{3}{2}}-36+36x^{\frac{5}{2}}+12x^{-1}-54x^{\frac{7}{2}}+9x^{-\frac{5}{2}}+27x^{\frac{9}{2}}+x^{-6}-6x^{-\frac{7}{2}})} = ? \quad \text{Ans. } 3x^{\frac{1}{2}}-2x^{\frac{1}{2}}+x^{-2}.$$

$$5. \sqrt[3]{(1+3x-5x^2+3x^3-x^4)} = ? \quad \text{Ans. } 1+x-x^2.$$

226. Proposition I.—*A Binomial can never be a perfect cube.*

Its cube root cannot be a monomial, for the cube of a monomial is a monomial. It cannot be a polynomial, for the cube of the simplest polynomial, viz.:—a binomial, has at least four terms.

Proposition II.—*A Trinomial can never be a perfect cube.*

Proposition III.—*A Quadrinomial is a perfect cube when, (arranged with respect to some letter), the first and last terms are perfect cubes, and the second and third are, respectively, three times the square of the cube root of each of these into the cube root of the other.*

227. III.—**To Deduce a Rule for the extraction of the n th root of polynomials.**

Let P represent the given polynomial, arranged according to the descending powers of any letter, and $r, r', r'',$ etc., the successive terms of the root, whatever form these terms may have. Suppose the root $(r+r'+r''+\dots)$ arranged according to the descending powers of the leading letter of P. Then r contains a higher power of the leading letter than r', r' than $r'',$ and so on. Let $s=r'+r''+r'''+\dots$. Then $P=(r+s)^n=r^n+nr^{n-1}s+\dots+s^n$. Since r contains a higher power of the leading letter than any other term of the root, r^n , the first term of the continued product $(r+r'+r''+\dots)(r+r'+r''+\dots)\dots\dots$, will contain a higher power of the leading letter than any other term of this product, and will therefore not reduce with any other. Since, then, the first term of the given arranged polynomial

was obtained, *without reduction*, by taking the n th power of the first term of the root,

IF WE EXTRACT THE n th ROOT OF THE FIRST TERM OF THE ARRANGED POLYNOMIAL, WE SHALL OBTAIN THE FIRST TERM OF THE REQUIRED ROOT.

Thus extracting the n th root of r^n we see that we have r , the first term of the root. We subtract, then, from the given polynomial, the n th power of r , in order to simplify our search for the other terms of the root, which terms can have nothing to do with the formation of r^n , as already explained. Let R_1 represent the remainder obtained by subtracting r^n from P . Then

$$\begin{aligned} R_1 &= P - r^n = nr^{n-1}s + n \frac{n-1}{2} r^{n-2}s^2 + \dots + s^n \\ &= nr^{n-1}(r' + r'' + r''' + \dots) + \dots \\ &= nr^{n-1}r' + nr^{n-1}r'' + \dots \end{aligned}$$

We can show, by reasoning precisely similar to that employed in (222) that $nr^{n-1}r'$ will be that term of the remainder that contains the highest power of the leading letter. It will therefore not reduce with any other term and will be found, *unreduced*, as the first term of our arranged remainder. Dividing $nr^{n-1}r'$ by nr^{n-1} we obtain r' , the second term of the root. Hence,

IF WE DIVIDE THE FIRST TERM OF OUR ARRANGED REMAINDER BY n TIMES THE $(n-1)$ th POWER OF THE FIRST TERM OF THE ROOT, WE SHALL OBTAIN THE SECOND TERM OF THE ROOT.

Now let $r + r'$, the sum of the terms of the root already found, be represented by a , and let s' represent the sum of those terms yet to be found. P will now equal $(a + s')^n = a^n + na^{n-1}s' + \dots + s'^n$. Subtracting a^n from P , we obtain, for the second remainder,

$$\begin{aligned} R_2 &= na^{n-1}s' + \dots + s'^n \\ &= n(r + r')^{n-1}(r'' + r''' + \dots) + \dots + s'^n \\ &= nr^{n-1}r'' + nr^{n-1}r''' + \dots + s'^n. \end{aligned}$$

On this remainder we may reason as already explained, and prove that $nr^{n-1}r''$ will appear, *without reduction*, as the first term of the arranged remainder, from which, by divid-

ing by nr^{n-1} , we may obtain r'' , the third term of the root. In a similar manner we may show that

THE FIRST TERM OF ANY REMAINDER, OBTAINED AS HERE INDICATED, WILL, UPON DIVISION BY n TIMES THE $(n-1)$ th POWER OF THE FIRST TERM OF THE ROOT, GIVE A NEW TERM OF THE ROOT.

As the letters employed may represent any quantities whatever, we deduce, for the extraction of the n th root of any polynomial, the following

RULE.—Arrange the polynomial according to the descending or ascending powers of some letter, and take the n th root of the first term for the first term of the n th root. Subtract the n th power of this term of the root from the given polynomial.

Divide the first term of the remainder by n times the $(n-1)$ th power of the first term of the root. The quotient will be the second term of the root. Subtract the n th power of that part of the root now found from the given polynomial.

Continue this process, using at each step the same divisor as before, until a remainder of zero is obtained, or, (if from the nature of the case, no such remainder can be obtained), as far as is desired.

Scholium I.—If a remainder of zero is obtained, the polynomial is a perfect n th power, of which the n th root is the sum of the terms found.

If from the nature of the case, a remainder of zero never can be obtained, the polynomial is an imperfect n th power, and an approximate root is found by continuing the indicated operation to any desired point.

Scholium II.—We see from the nature of the reasoning that the polynomial might have been arranged according to the ascending powers of any letter, and the root found as indicated in the rule.

Scholium III.—By considering r, r', r'' , etc., as representing figures expressing units of different orders, we may apply the principles just discussed to numbers. The extraction of the n th root of numbers is, however, so important an operation that its discussion is reserved for a subsequent article.

Scholium IV.—From Arts. 201-5 we see that the n th root of the given polynomial may also be obtained by the binomial formula, the exponent of the given polynomial being $\frac{1}{n}$.

228. Corollary.—*The first term of the first remainder found in the process of extracting any root of a polynomial is always the same as the second term of the given polynomial.*

229. Proposition I.—*A Binomial is never a perfect power.*

For any power of a monomial is a monomial, and the n th power of the simplest polynomial (a binomial) must (203) have at least $n+1$ terms.

Proposition II.—*In order that a polynomial should be a perfect power of any degree, it must have at least as many terms as there are units in the exponent of the power, plus 1.*

For, the n th power of a monomial is always a monomial, and the n th power of the simplest polynomial has (203) $n+1$ terms; hence a polynomial of n terms or less cannot be a perfect n th power.

If it be required to extract the n th root of a polynomial that is an imperfect n th power, though the root can never be obtained exactly, it may be obtained to within any desired degree of approximation by performing the operations indicated in the foregoing rules until a sufficient number of terms of the root has been obtained.

EXAMPLES.

1. Extract the fifth root of

$$\begin{array}{r} 32x^{10} + 240x^8y^2 + 720x^6y^4 + 1080x^4y^6 + 810x^2y^8 + 243y^{10} \quad | 2x^2 + 3y^2 \\ \hline 32x^{10} \end{array}$$

$$\hline 240x^8y^2 + 720x^6y^4 + \dots \dots \dots (1)$$

$$\begin{array}{r} 32x^{10} + 240x^8y^2 + 720x^6y^4 + 1080x^4y^6 + 810x^2y^8 + 243y^{10} \\ \hline 0 \dots \dots \dots (2) \end{array}$$

The polynomial being already arranged according to the descending powers of x , the fifth root of the first term, $\sqrt[5]{32x^{10}} = 2x^2$, is the first term of the root, and $240x^8y^2$ is the first term of the first remainder. Hence $240x^8y^2 \div 5(2x^2)^4 = 3y^2$ is the second term of the root, and as $(2x^2 + 3y^2)^5$ equals the given polynomial, the second remainder is 0, and $2x^2 + 3y^2$ is the required root.

$$2. \sqrt[6]{1-6a^{\frac{1}{3}}+15a^{\frac{2}{3}}-20a+15a^{\frac{4}{3}}-6a^{\frac{5}{3}}+a^2}=? \quad \text{Ans. } 1-a^{\frac{1}{3}}.$$

$$3. \sqrt[4]{1-4a^{\frac{1}{3}}+6a-4a^{\frac{2}{3}}+a^2}=? \quad \text{Ans. } 1-a^{\frac{1}{3}}.$$

4.

$$\sqrt[3]{8x^6-36c^1x^{\frac{5}{2}}+102c^2x^{\frac{4}{2}}-171c^3x^{\frac{3}{2}}+204c^4x^{\frac{2}{2}}-144c^5x^{\frac{1}{2}}+64c^6}$$

$$\text{Ans. } 2x^{\frac{2}{2}}-3c^1x^{\frac{1}{2}}+4c^2.$$

$$5. \text{ Extract the fifth root of } x^{-20}+15x^{-16}-5x^{-12}+90x^{-12}-60x^{-10}+280x^{-8}-70x^{-6}+49x^{-4}-550x^{-2}+495-513x^2+275x^4-90x^6+15x^8-x^{10}.$$

$$\text{Root } x^{-4}+3-x^2.$$

$$1^\circ \text{ Remainder, } 15x^{-16}-\dots\dots$$

$$(x^{-4}+3)^5=x^{-20}+15x^{-16}+90x^{-12}+270x^{-8}+405x^{-4}+243.$$

$$2^\circ \text{ Remainder, } -5x^{-12}-\dots\dots$$

$$(x^{-4}+3-x^2)^5=\text{given polynomial.}$$

$$6. \sqrt[7]{(128a^{14}-1344a^{12}b^{\frac{2}{7}}x^{\frac{2}{7}}+6048a^{10}b^{\frac{4}{7}}x^{\frac{4}{7}}-15120a^8b^{\frac{6}{7}}x^{\frac{6}{7}}+22680a^6b^{\frac{8}{7}}x^{\frac{8}{7}}-20412a^4b^{\frac{10}{7}}x^{\frac{10}{7}}+10206a^2b^{\frac{12}{7}}x^{\frac{12}{7}}-2187b^{\frac{14}{7}}x^2)}=?$$

$$\text{Ans. } 2a^2-3b^{\frac{2}{7}}x^{\frac{2}{7}}.$$

$$7. \sqrt[3]{\left(x^3-\frac{1}{x^3}-3x+\frac{3}{x}\right)}=?$$

$$\sqrt[3]{\left(\frac{4}{a}-12b^{\frac{4}{3}}+4ab^{-\frac{1}{3}}+9ab^{\frac{2}{3}}-6a^2b+a^3b^{-\frac{2}{3}}\right)}=?$$

$$\text{Ans. } x-\frac{1}{x}; 2a^{-\frac{1}{3}}-3a^{\frac{1}{3}}b^{\frac{4}{3}}+a^{\frac{2}{3}}b^{-\frac{1}{3}}.$$

$$8. \sqrt[3]{1-x}=? \quad \text{Ans. } 1-\frac{x}{3}-\frac{x^2}{9}-\frac{5x^3}{81}-\frac{10x^4}{243}-\dots$$

$$9. \sqrt[7]{\left(\frac{35}{2}x^4y^2+168x^2y^4+\frac{1}{128}x^7+\frac{21}{8}x^5y^2+70x^3y^4+128y^7+224xy^6+\frac{7}{32}x^2y\right)}=? \quad \sqrt[5]{1-x^5}=?$$

$$10. \sqrt[4]{\frac{1}{x^3}-\frac{4}{x^2y}+\frac{6}{x^{\frac{3}{2}}y^2}-\frac{4}{x^{\frac{1}{2}}y^3}+\frac{1}{y^4}}=? \quad \text{Ans. } x^{-\frac{3}{4}}-y^{-1}.$$

The solution of examples like the last, in which the index of the required root is composite, is often simplified by the application of the principle

APPROXIMATE ROOTS.

231. The binomial formula furnishes a method of finding, to any desired degree of accuracy, the roots of polynomials that are imperfect powers of the degree indicated. Thus, if $\sqrt[n]{a+b}$ is required, we may place $\sqrt[n]{a+b}=(a+b)^{\frac{1}{n}}$ and develop by the formula, finding as many terms of the expansion as may be necessary or expedient in the particular case. Similarly $\sqrt[n]{\frac{1}{a^2-c^2}}$ equals $(a^2-c^2)^{-\frac{1}{n}}$, $\sqrt[n]{a+b+c}=(a+b+c)^{\frac{1}{n}}$, $\sqrt[n]{(a+b+c+d)^r}=(a+b+c+d)^{\frac{r}{n}}$, etc.

232. Approximate Roots of Fractions.—If both terms of a fraction the n th root of which is to be extracted are perfect n th powers, the application of the rule (215) gives the exact n th root at once. If, however, either or both terms be imperfect n th powers, the n th root of the fraction cannot be obtained exactly, for by (214) the n th root of the imperfect powers in the numerator and denominator cannot be expressed in exact parts of 1. As the numerator or denominator of the n th root cannot be exactly expressed, the n th root cannot be exactly expressed. We can in this case approximate as closely as we please to the true root; in other words, we may obtain a quantity that differs from the true root by as small a quantity as we please.

233. I.—To Deduce a Rule for extracting the n th root of a fraction to within less than the fractional unit.

Let $\frac{a}{b}$ be the given fraction. If, now, b were a perfect n th power, we should be able to extract its n th root. As we can add or subtract fractions only when they have a common denominator, we can determine the degree of approx-

imation in our result only if its denominator be b , as it is to differ from the true root by less than $\frac{1}{b}$.

We can make the denominator a perfect n th power by multiplying it by b^{n-1} . We must also multiply the numerator by b^{n-1} , so that the value of the fraction may not be changed. Multiplying, we obtain $\frac{ab^{n-1}}{b^n}$. Let r^n be the greatest perfect n th power in ab^{n-1} . Then $\frac{ab^{n-1}}{b^n}$ will be greater than $\frac{r^n}{b^n}$ and less than $\frac{(r+1)^n}{b^n}$, and the n th root of $\frac{ab^{n-1}}{b^n}$ will be greater than the n th root of $\frac{r^n}{b^n}$ and less than the n th root of $\frac{(r+1)^n}{b^n}$. Hence $\sqrt[n]{\frac{ab^{n-1}}{b^n}} > \frac{r}{b}$ and $< \frac{r+1}{b}$. The difference, then, between $\sqrt[n]{\frac{ab^{n-1}}{b^n}}$ and either $\frac{r}{b}$ or $\frac{r+1}{b}$ will be less than the difference between $\frac{r+1}{b}$ and $\frac{r}{b}$. $\frac{r+1}{b} - \frac{r}{b} = \frac{1}{b}$. Hence the difference between the true n th root of $\frac{ab^{n-1}}{b^n}$ and $\frac{r}{b}$ or $\frac{r+1}{b}$ is less than $\frac{1}{b}$. In other words, $\frac{r}{b}$ or $\frac{r+1}{b}$ is the n th root of $\frac{ab^{n-1}}{b^n} = \frac{a}{b}$, to within less than $\frac{1}{b}$. As a , b and n may represent any quantities whatever, we deduce the following

RULE.—Multiply the numerator of the given fraction by the $(n-1)$ th power of the denominator. Extract the n th root of the product and divide this root by the denominator of the given fraction.

The rule of Article 223 enables us to extract the n th root of an entire quantity to within less than 1, the unit of entire or integral quantities, and the rule of Article 233 enables us to extract the n th root of a fraction to within less than the unit of the fraction. We pass now to the

233*—II. General Case.—*To Deduce a Rule for extracting the n th root of any quantity to within less than any given fraction.* Let Q represent the given quantity, and $\frac{r}{s}$ the fraction indicating the degree of approximation. $\frac{r}{s} = \frac{1}{\frac{s}{r}}$. Let v represent the value of the fraction $\frac{s}{r}$;

then $\frac{r}{s} = \frac{1}{v}$, and we are required to extract the n th root of Q to within $\frac{1}{v}$. $Q = \frac{Qv^n}{v^n} = \frac{Qv^n}{v^n}$. If r^n equals the greatest perfect n th power in Qv^n , $\frac{Qv^n}{v^n}$ lies between $\frac{r^n}{v^n}$ and $\frac{(r+1)^n}{v^n}$. (Since $Qv^n > r^n$ and $< (r+1)^n$).

Hence the n th root of $\frac{Qv^n}{v^n}$ is greater than the n th root of $\frac{r^n}{v^n}$ and less than the n th root of $\frac{(r+1)^n}{v^n}$. Hence the n th root of $\frac{Qv^n}{v^n}$ lies between $\frac{r}{v}$ and $\frac{r+1}{v}$, that is, the difference between the n th root of $\frac{Qv^n}{v^n}$ and $\frac{r}{v}$ or $\frac{r+1}{v}$ is less than the difference between $\frac{r+1}{v}$ and $\frac{r}{v}$. This difference $(\frac{r+1}{v} - \frac{r}{v})$ is $\frac{1}{v}$. Hence, either $\frac{r}{v}$ or $\frac{r+1}{v}$ differs from the true root of $\frac{Qv^n}{v^n}$ by less than $(\frac{1}{v} = \frac{r}{s})$. Hence either $\frac{r}{v}$ or $\frac{r+1}{v}$ is the root required. Since our reasoning is perfectly general, we deduce the

RULE.—*Multiply the given quantity by the n th power of the value of the reciprocal of the given fraction. Extract the n th root of the product to within 1, and divide the result by the reciprocal of the given fraction.*

Scholium.—The rule assumes that the denominator of the given quantity is 1, i. e., the quantity is in the form of an integer. If the given quantity is a fraction, it may be reduced to this integral form by transferring all the factors of the denominator to the numerator; thus $\frac{x}{y} = xy^{-1}$, $\frac{E^2}{F^3} = E^2F^{-3}$, and so on.

EVOLUTION OF NUMBERS.

234. To Deduce a Rule for extracting any root of a given number.

Let m be the index of the required root.

Let us form a table of the m th powers of the first ten

	1	2	3	4	5	6	7	8	9	10
--	---	---	---	---	---	---	---	---	---	----

numbers thus: $1^m 2^m 3^m 4^m 5^m 6^m 7^m 8^m 9^m 10^m$. Every power of 1 is 1; hence $1^m=1$. Any power of 10 is 1 followed by as many zeroes as there are units in the exponent of the power. Thus $10^2=100$, $10^3=1000$, and so on. Hence 10^m will be 1 followed by m zeroes; it will be expressed, then, by $m+1$ figures. All the numbers from 1^m up to 9^m inclusive, will therefore be expressed by less than $m+1$ figures, for 1 followed by m zeroes is the smallest number expressed by $m+1$ figures. If, then, we are required to find the m th root of a number expressed by less than $m+1$ figures, we look in the lower line of the above table. If the number given occurs in this line, it will be a perfect m th power, and its m th root will be the number immediately above it, the corresponding number in the upper line. If the given number does not occur in the lower line of the table, it is an imperfect m th power and falls between two numbers in the lower line, its m th root falling between the two corresponding numbers in the upper line, either of which will be the m th root to within less than 1. If the number be greater than 1 followed by m zeroes, it will not be found in the above table, and its root will exceed 10; i. e., will contain tens and units. If, now, we let t represent the tens in the root, and u the units, the root will be $t+u$, and the m th power of this root, i. e., the given number, will be $(t+u)^m$; or, by the binomial formula, $(t+u)^m=t^m+m t^{m-1}u+\dots+u^m=N$ (the given number being represented by N). Since the m th power of one ten, the smallest number of tens that can occur, is 1 followed by m zeroes, t^m , the m th power of the tens of N , will be at least 1 followed by m zeroes; i. e., it will certainly have m zeroes for the m right-hand figures. Hence we point off the first m figures on the right of the given number, as having nothing to do with the m th power of the tens, which must

be found in the figures to the left of the point. Let us suppose, for simplicity, that there are not more than m figures to the left of the point, so that the m th root of the greatest perfect m th power in these figures may be less than 10. The m th root, then, of the greatest perfect m th power in these figures is the number of tens in the root. Having thus obtained t , we subtract its m th power from N to simplify our search for the rest of the root. We thus obtain

$$N - t^m = mt^{m-1}u + m \frac{m-1}{2} t^{m-2}u^2 + \dots + u^m.$$

If now we can pick out the part $mt^{m-1}u$, it is obvious that we may, by dividing it by mt^{m-1} , obtain u , the units of the root. We do not know how to separate $mt^{m-1}u$ completely from the other terms, but we do the best that we can in this direction. Since one ten, the first power of 10, has one zero on the right, the $(m-1)$ th power of 10 will have $m-1$ zeroes on the right. Hence t^{m-1} must have $m-1$ zeroes as the right-hand figures, and therefore the $m-1$ figures on the right of our remainder cannot have anything to do with $mt^{m-1}u$. This term will then be found in the part of the remainder obtained from the first left-hand period, with one figure from the next or right-hand period. We shall call this remainder, leaving out of consideration the $m-1$ figures on the right, the trial dividend. As the square of the units may be expressed by two figures, and t^{m-2} will be at least 1 followed by $m-2$ zeroes, it is possible that a part of the term $m \cdot \frac{m-1}{2} t^{m-2}u^2$ may be found in the trial dividend, and for similar reasons part of the next term may be found in the trial dividend. The trial dividend, then, contains the *whole* of the term $mt^{m-1}u$, and *may* contain, in addition, parts of other terms. The separation of what we have called the trial dividend is, however, the best that we can do toward segregating $mt^{m-1}u$. Had we separated it wholly from the other terms, the division of it by mt^{m-1} would give u , but as we have not separated it wholly from the other terms, but as nearly so as possible,

the division of our trial dividend by mt^{m-1} will give u either exactly, or nearly so. As the trial dividend must always contain *at least* $mt^{m-1}u$, the quotient obtained by dividing it by mt^{m-1} must be *at least as large as* u ; and as parts of other terms may occur in the trial dividend, the quotient figure may be too large. If now we raise the root found to the m th power, and find this result greater than N , we know that the quotient figure just spoken of is too large. Hence, we diminish the number of units in the root until the m th power of the root found does not exceed the given number. If the m th power of the root is then precisely equal to the given number, N is a perfect m th power, and the root found is its m th root. If the m th power of the root found is less than N , the given number is an imperfect m th power, and the root found is correct to within less than 1. For, as just explained, the units figure cannot be too small, and if we increase it by 1, the m th power of the resulting number will be greater than N . Hence N lies between the m th power of the root found and the m th power of the number greater by 1 than this root, *i. e.*, the root found differs from the true m th root of N by less than 1.

If, after pointing off the m right-hand figures, the part on the left should contain more than m figures, the number of tens, *i. e.*, the m th root of the greatest perfect m th power therein, would be expressed by more than one figure, that is, the root would contain hundreds. To get the number of tens we should then be obliged to reason on this left-hand part as we have here reasoned on N , pointing off m figures to the right, and so on.

In general, as the m th power of 10 is 1 followed by m ciphers, the m th power of 100 1 followed by $2m$ ciphers, the m th power of 1000 1 followed by $3m$ ciphers, and so on, the m th power of any number between 10 and 100 cannot have more than $2m$ figures, the m th power of any number between 100 and 1000 (*i. e.*, expressed by 3 figures) cannot

have more than $3m$ figures, and so on. Hence, dividing any number into periods of m figures each, beginning at the right, the number of periods (counting the extreme left-hand portion as a period, even if it contain less than m figures), will equal the number of orders of units in the root.

From the foregoing considerations we deduce, for the extraction of the m th root of a number, the following

RULE.—Separate the number into periods of m figures each, beginning at the right. The left-hand period may contain less than m figures. Extract the m th root of the greatest perfect m th power in the left-hand period, for the first figure of the root. Subtract this perfect m th power from the left-hand period, and bring down the first figure of the second period. Divide this result by m times the $(m-1)$ th power of the first figure of the root. The quotient will be the next figure of the root, or something greater.

Find the m th power of that part of the root now obtained. If this be greater than the first two left-hand periods, diminish the figure last found until the m th power of the part of the root found does not exceed the two left-hand periods.

Then, subtract the m th power of that part of the root now found from the two left-hand periods, and to the remainder bring down the first figure of the next period. Divide this result by m times the $(m-1)$ th power of that part of the root already found, and continue the operations above indicated until all the periods have been operated upon.

EXAMPLES.

1. Extract the 7th root of 19891027786401117.

$$\begin{array}{r}
 198^{\circ}9102778^{\circ}6401117 \quad |213 \\
 128 \qquad \qquad \qquad 7.(2)^6=448 \\
 \hline
 709 \dots\dots\dots (1)1^{\circ} \text{ trial dividend.} \\
 1801088541 \dots\dots\dots (21)^7 \\
 \hline
 1880142376 \dots\dots\dots (2)2^{\circ} \text{ trial dividend.} \\
 \qquad \qquad \qquad 7.(21)^6=7 \times 85766121=600362847 \\
 19891027786401117 \dots\dots\dots (213)^7
 \end{array}$$

Pointing off, from the right, periods of 7 figures each, we find the greatest perfect 7th power in the left-hand period to be 128; as $\sqrt[7]{128}=2$, 2 is the first figure of the root. Subtracting 2^7 from the first period, and to the result annexing 9, the first figure of the second period, we have 709 as the first trial dividend. $709 \div 448$ gives 1 as the second figure of the root, and subtracting $(21)^7$ from the first two periods, we obtain 188014237, to which annexing the first figure of the third period we have 1880142376 as the second trial dividend, which, divided by $7(21)^6=600362847$, gives 3 as the third figure of the root. As $(213)^7$ equals the given number, the last remainder is 0, and 213 is the root required.

2. $\sqrt[3]{1231171548132409344}=?$ Ans. 384.

3. $\sqrt[5]{36936242722357}=?$ $\sqrt[5]{5550000000000000}=?$

[illegible]

5. $\sqrt[8]{1968300000000000000}=?$ $\sqrt[8]{279726264000}=?$

6. $\sqrt[6]{3462825991689} = ?$

235. Scholium I.—The extraction of the square root of numbers is a special case, falling under (234). Here $m=2$ and $m-1=1$. Instead, however, of subtracting at each step in the process the square of the whole root already found, it is usually considered easier to subtract from the remainder that part of the square of the tens plus the units not already subtracted, viz., twice the product of the tens by the units, plus the square of the units, on the principle indicated in (218).

We thus have the following

RULE.—Separate the given number into periods of two figures each, beginning at the right; the extreme left-hand period may contain but one figure.

Extract the square root of the greatest perfect square in the left-hand period for the first figure of the root. Subtract this perfect square from the left-hand period, and bring down the next period.

Divide this result (not considering the last figure), by twice the root already found. Annex the quotient to the root and to the divisor. Multiply the augmented divisor by the second figure of the root and subtract the product from the first remainder. If the product should be greater than the first remainder, diminish the number of units in the root until this product is no longer greater than the first remainder, and then subtract. Bring down the next period.

With twice the root already found as a trial divisor, proceed as before until all the periods have been operated upon.

236. *The number of digits in the square root of a number being $2n+1$, after $n+1$ have been obtained by (235) the other n may be found by simple division.*

For, N being the given number, r the part of the root found by (235) and x the part expressed by the last n digits,

$N = (r+x)^2 = r^2 + 2rx + x^2$. $\therefore N - r^2 = 2rx + x^2$ and $\frac{N-r^2}{2r} = x + \frac{x^2}{2r}$; since if equals be divided by the same quantity, the quotients will be equal. By supposition r is the part of the root to the left of the last n figures, and as it contains $n+1$ digits it cannot be less than 10^{2n} ; x contains only n digits, and therefore x^2 cannot contain more than $2n$ digits, and must be less than 10^{2n} . Hence $x^2 < 2r$ and $\frac{x^2}{2r}$ must be a proper fraction, i. e., must be less than 1. As the indicated division gives x and the remainder $\frac{x^2}{2r} < 1$, this division gives x , the latter part of the root, to within 1.

EXAMPLES.

Extract the square root of the following numbers:
 277729; 4222140484; 44105040144; 3518743761; 100020001;
 3737373414351721; 186711476424322756; 54782211136;
 32239684.

237. Scholium II.—If we are required to extract the cube root of a number, $m=3$, $m-1=2$, and the rule of Art. 234 becomes the following

RULE.—*Separate the number into periods of three figures each, beginning at the right. The left-hand period may contain less than three figures. Extract the cube root of the*

greatest perfect cube in the left-hand period, for the first figure of the root.

Subtract this perfect cube from the left-hand period, and bring down the first figure of the second period. Divide this result by three times the square of the part of the root already found. The quotient will be the next figure of the root, or something greater.

Find the cube of that part of the root now obtained. If this be greater than the first two left-hand periods of the given number diminish the quotient last found until the cube of the part of the root obtained does not exceed the two left-hand periods.

Subtract the cube of that part of the root now found from the two left-hand periods and to the remainder bring down the first figure of the next period. Divide this result by three times the square of that part of the root already found, and proceed as before, continuing the operation until all the periods have been operated upon.

238. *The number of digits in the cube root of a number being $2n+2$, and $n+2$ having been obtained by (237), the other n may be obtained by division only.*

For, N being the given number, c , the part of the root expressed by the first $n+2$ figures, and x the other part, $N=(c+x)^3=c^3+3c^2x+3cx^2+x^3$.

$N-c^3=3c^2x+3cx^2+x^3$ and $\frac{N-c^3}{3c^2}=x+\frac{x^2}{c}+\frac{x^3}{3c^2}$; since equals, divided by the same quantity, give equal quotients. Now $x < 10^n$ since x contains less than $n+1$ digits, and c is at least 10^{2n+1} . Hence, x^3 being less than 10^{3n} , $\frac{x^2}{c} < \frac{10^{2n}}{10^{2n+1}} \cdot \frac{x^2}{c} < \frac{1}{10}$.

Further, $x^3 < 10^{3n}$, since $x < 10^n$, and c^3 is at least as large as 10^{4n+2} since c is at least as great as 10^{2n+1} . Therefore $\frac{x^3}{3c^2} < \frac{10^{3n}}{3 \times 10^{4n+2}}$

$$\frac{10^{3n}}{3 \times 10^{4n+2}} = \frac{1}{3 \times 10^{n+2}} \cdot \frac{x^3}{3a^2} < \frac{1}{3 \times 10^{n+2}};$$

consequently $\frac{x^2}{c} + \frac{x^3}{3c^2} < \frac{1}{10} + \frac{1}{3 \times 10^{n+2}}$, and is therefore less than 1.

Hence as the division indicated above gives x , the part of the root ex-

pressed by the last n figures, with a remainder less than $\frac{1}{10} + \frac{1}{3 \times 10^{n+2}}$, this division gives the part of the root mentioned to within 1.

EXAMPLES.

Extract the cube root of the following numbers: 39651821; 44361864; 238501021853375; 33199964344; 403583419; 115501303; 337153536; 34012224000000; 183056926752.

239. Scholium III.—*To apply principles concerning the n th root of quantities to a root having any given numerical index, simply substitute this index for n in the statement or demonstration of these principles.*

240. To Deduce a Rule for extracting any root of a Numerical Fraction.

Since the reasoning of Art. 215 is perfectly general, it applies to the case of numerical quantities. Hence, to extract any root of a numerical fraction, we have the following

RULE.—*Extract the required root of the numerator for the new numerator, and the required root of the denominator for the new denominator.*

EXAMPLES.

$$\sqrt[6]{\frac{4096}{15625}} = ? \quad \sqrt[9]{\frac{512}{19683}} = ? \quad \sqrt[3]{\frac{39651821}{163667323}} = ? \quad \sqrt[7]{\frac{16384}{1801088541}} = ?$$

241. Corollary—*To extract the square root of a numerical fraction extract the square root of the terms for the terms of the new fraction. To extract the cube root of a numerical fraction extract the cube root of the terms, and so on.*

EXAMPLES.

$$\begin{array}{cccc} \sqrt{\frac{576}{676}} = ? & \sqrt{\frac{289}{784}} = ? & \sqrt{\frac{324}{841}} = ? & \sqrt{\frac{729}{1331}} = ? \\ \sqrt[3]{\frac{9261}{10648}} = ? & \sqrt[3]{\frac{19683}{21952}} = ? & & \end{array}$$

242. The rule of (240) gives us the exact root only when the numerator and denominator of the given fraction are both perfect powers of the given degree. If the terms are not perfect powers of the indicated degree, we may approximate to the true root to any desired degree of accuracy by the principles of Arts. 233-3*. Hence the following rules:

I. To extract the square root of any Numerical Fraction to within less than the fractional unit.

Multiply the numerator by the denominator, extract the square root of the product to within 1, and divide this result by the denominator.

II. To extract the cube root of any Numerical Fraction to within less than the fractional unit.

Multiply the numerator by the square of the denominator. Extract the cube root of the product to within 1, and divide by the denominator.

III. In general, the rules of Articles 233 and 234 apply verbatim to numerical quantities.

EXAMPLES.

1. Extract the square root of $\frac{5}{7}$ to within $\frac{1}{7}$. *Ans.* $\frac{6}{7}$.

Find

2. $\sqrt{7}$ to within $\frac{1}{7}$; $\sqrt{8}$ to within $\frac{1}{9}$; $\sqrt{7}$ to within $\frac{1}{2}$.

Ans. $2\frac{5}{7}$; $2\frac{8}{9}$; $2\frac{1}{2}$.

3. $\sqrt[3]{5}$ to within $\frac{1}{5}$; $\sqrt[3]{-10}$ to within $-\frac{1}{7}$; $\sqrt{\frac{7}{6}}$ to within $\frac{1}{5}$.

Ans. $1\frac{3}{5}$; $-2\frac{1}{7}$; 1.

4. $\sqrt[3]{8}$ to within $\frac{1}{100}$; $\sqrt[9]{27}$ to within $\frac{1}{100}$; $\sqrt[3]{473}$ to within $\frac{1}{20}$

Ans. 1.26; 1.74; $7\frac{3}{4}$.

$$5. \sqrt[3]{\frac{14}{25}} \text{ to within } \frac{1}{100}; \sqrt[3]{15} \text{ to within } \frac{1}{12}.$$

DECIMALS.

243. To Deduce a Rule for extracting any root of a Decimal Fraction.

Since the number of zeroes in the denominator of a decimal fraction is always equal to the number of decimal places in the numerator as written, and since any power of 10 is 1 followed by as many zeroes as there are units in the exponent of the power, we can always tell at a glance whether the denominator of a given decimal is a perfect power of the indicated degree, or not. We will take up the discussion as follows:

I. Both terms are perfect powers of the degree indicated.

Expressing the denominator, we can apply directly the rule of Article 215. The result will be the n th root of the numerator divided by the n th root of the denominator. Since the n th power of 10 is 1 followed by n zeroes, the n th root of 1 followed by any number of zeroes, is 1 followed by $\frac{1}{n}$ th as many zeroes. If we write the n th root of the fraction, obtained as just described, as a decimal, the number of decimal places will equal the number of zeroes in the new denominator, i. e., $\frac{1}{n}$ th as many as there were in the original denominator. The number in the original denominator equals the number of decimal places in the decimal as originally written. Hence the number of decimal places in the root is $\frac{1}{n}$ th of that in the given decimal.

TO ILLUSTRATE. $\sqrt[3]{.729}=?$ $\sqrt[3]{.729}=\sqrt[3]{\frac{729}{1000}}=\frac{9}{10}=.9$. Here the degree of the root is 3. $n=3$, $\frac{1}{n}=\frac{1}{3}$, and the number of decimal places in the root is one third the number in the given fraction.

From the foregoing considerations we deduce the following

RULE.—*Extract the required root of the given decimal as if it were an integer. Point off in the root the number of decimal places in the given fraction divided by the number of units in the index of the required root.*

EXAMPLES.

$$\sqrt[4]{.024389}=? \quad \sqrt[4]{.019683}=? \quad \sqrt[6]{.004096}=? \quad \sqrt[4]{.000729}=?$$

$$\sqrt[5]{.0000000243}=? \quad \sqrt[7]{.0000128}=? \quad \sqrt[8]{.0000000000000256}=?$$

$$\sqrt[6]{.000093385106978409}=? \quad \sqrt[4]{.000009261}=?$$

$$\sqrt[4]{.000000000000221445125}=?$$

$$\sqrt[4]{.000000000000221445125} = \sqrt[4]{\frac{221445125}{1000000000000000000}}$$

$$\frac{605}{1000000} = .000605; \text{ or } \sqrt[4]{221445125} = 605, \text{ and pointing off } 7 \left(= \frac{21}{3} \right) \text{ decimal places we have } .000605 \text{ Ans.}$$

II. The numerator is an imperfect power of the degree indicated; the denominator is a perfect power of that degree.

In this case, if we extract the required root of the numerator to within 1, and divide the result by the required root of the denominator, this result will, by principles already discussed, be the required root of the fraction to within 1 divided by the required root of the denominator. Dividing the required root of the numerator by the required root of the denominator, is, however, the same as pointing off as many decimal places from the right of the root as there are zeroes in the root of the denominator. This number of zeroes will, as already explained, always equal the number of places in the given decimal, divided by the index of the required root.

$$\text{TO ILLUSTRATE: } \sqrt[3]{.692} = \frac{\sqrt[3]{692}}{\sqrt[3]{1000}} = \frac{8}{10}, \text{ to within less than}$$

$\frac{1}{10}$. Here $n=3$, $\frac{1}{n}=\frac{1}{3}$. We could evidently have obtained this result by extracting the cube root of 692 (as an integer), to within 1, and pointing off in the result one-third as many decimal places as there are places in .692.

From the foregoing considerations we deduce the

RULE.—*Extract the required root of the decimal to within 1, as if it were integral. Point off in the root a number of decimal places equal to the number in the given decimal, divided by the index of the root; the result will be the required root to within 1 divided by the indicated root of the denominator.*

EXAMPLES.

$$\sqrt[2]{.653272}= ? \quad \sqrt[4]{.52763415}= ? \quad \sqrt[3]{.152673}= ? \quad \sqrt[5]{.000000231}= ?$$

If the approximation to the degree indicated in the rule is not sufficiently close, we may annex 0's to the numerator, provided we annex the same number to the denominator, and provided, also, that the number of 0's annexed is a multiple of the index of the root. This operation, while not affecting the value of the fraction, still keeps the denominator a perfect power of the degree indicated, and therefore does not affect the applicability of the rule.

Thus let $\sqrt[6]{.000000024389}$ be required, to within .0001; the application of the rule to the fraction as proposed would give the root only to within .01. Now, $\frac{24389}{1000000000000} = \frac{2438900000000000}{100000000000000000000}$. The sixth root of the numerator is 538 to within 1; the sixth root of the denominator is 10000. Hence the 6th root of the fraction is $\frac{538}{10000} = .0538$, to within $\frac{1}{10000} = .0001$. This result might evidently have been obtained at once by annexing 0's to the numerator, and proceeding as before indicated.

III. The denominator is an imperfect power of the degree indicated.

In this case, if we can transform the fraction so as to make the denominator a perfect power of the indicated

degree, we can obtain the indicated root by **I** or **II**. Any power of 10 being 1 followed by as many zeroes as there are units in the exponent of the power, any power of 100 will be 1 followed by twice as many zeroes, any power of 1000 will be 1 followed by thrice as many zeroes, and so on. Hence if we can make the number of zeroes in the denominator a multiple of the index of the required root, the denominator will be a perfect power of the degree indicated, and the rules of **I**, **II** may be followed. We can annex zeroes to the denominator, if we annex the same number to the numerator. The denominator not being written, we determine the number of zeroes that would enter it if it were written by the number of figures in the numerator. If, then, we annex to the numerator, *i. e.*, to the decimal as written, as many zeroes as are necessary to make the whole number of decimal places a multiple of the index of the root, we may apply the rules of **I** or **II**.

TO ILLUSTRATE: $\sqrt[3]{.00173} = \sqrt[3]{\frac{173}{100000}}$. 100000 is not a perfect cube, but by annexing one zero to each term we obtain $\frac{1730}{1000000}$, in which the denominator is a perfect cube. $\sqrt[3]{\frac{1730}{1000000}} = \frac{12}{100}$, to within $\frac{1}{100}$. Hence $\sqrt[3]{.00173} = 0.12$, to within 0.01. This result might evidently have been obtained by annexing to .00173 one zero, making .001730, then extracting the cube root as if the number were integral, and pointing off in the result one third as many places as there are decimal places in the transformed decimal.

It will be seen that the number of decimal places in the result indicates the degree of approximation. (Thus in the example, the root is true to within .01). Hence, in annexing zeroes, we may annex a sufficient number to make the approximation as close as we please.

From the foregoing considerations we deduce the

RULE.—To extract the *n*th root of a Decimal.

Annex as many zeroes as may be necessary to render the whole number of decimal places a multiple of n. If a certain number of places be required in the root, annex zeroes until the whole number of decimal places is n times the number required in the root.

Extract the required root of the decimal as if the decimal were an integer. Point off, in the root, a number of decimal places equal to the number in the prepared decimal, divided by the index of the root.

244. Scholium.—*The rule just given evidently applies to the extraction of any root of ANY DECIMAL WHATEVER, to within any given limit.*

245. Corollary.—*To extract any root of a Common Fraction to any number of decimal places, i. e., when the degree of approximation is indicated by a decimal, we have, from the principles now discussed, the following*

RULE.—*Reduce the common fraction to a decimal, continuing the division until the number of decimal places is n times the number required in the root, of which n is the index. Extract the root as in the case of integers, and point off in the root the required number of decimal places.*

EVOLUTION OF MIXED NUMBERS.

246. To Deduce a Rule for extracting any root of a Mixed Number.

Since the reasoning of Article 216 is perfectly general, it applies directly to the case of mixed numbers. Hence we may reduce the given number to an equivalent fraction, and then extract the root, either exact or approximate, of this fraction, by the rules for the evolution of fractions. Hence the following

RULE.—*Reduce the mixed number to an equivalent fraction. Extract the required root of this fraction by the rules for evolution of fractions. Reduce the result to a mixed number.*

247. Corollary.—*To extract the n th root of a number partly integral and partly a decimal fraction, we have the*

RULE.—*Annex zeroes until the whole number of decimal places is n times the number required in the root. Extract the n th root as if the number thus obtained were integral. Point off in the root the required number of decimal places.*

EXAMPLES.

1. Find
- $\sqrt[3]{.02}$
- to within .00001

$$\begin{array}{r}
 .0200000000000000 \overline{)27144} \\
 \underline{8} \\
 12 \overline{)120} \dots 1^{\circ} \text{ Trial Dividend} \\
 \underline{19683} \dots = (27)^3 \\
 2187 \overline{)3170} \dots 2^{\circ} \text{ Trial Dividend} \\
 \underline{19902511} \dots = (271)^3 \\
 220323 \overline{)974890} \dots 3^{\circ} \text{ Trial Dividend} \\
 \underline{19990770344} \dots = (2714)^3 \\
 22197388 \overline{)92296560} \dots 4^{\circ} \text{ Trial Dividend.} \\
 \underline{19999610601984} \dots = (27144)^3
 \end{array}$$

Annexing zeroes until the whole number of decimal places is $5 \times 3 = 15$, and extracting the cube root of the resulting number we obtain 27144 as this cube root to within 1. The cube root of the transformed denominator being 100000, the cube root of the fraction is $\frac{27144}{100000} = .27144$, which result might have been obtained at once by pointing off $\frac{15}{3} = 5$ decimal places, in accordance with the rule.

- 2.
- $\sqrt[3]{.555}=?$
- $\sqrt[3]{.4}=?$
- $\sqrt[3]{.1}=?$
- $\sqrt[3]{.27}=?$
- $\sqrt[3]{.005}=?$

Ans. .82179; .63246; .46416; .64633; .07071.

- 3.
- $\sqrt[3]{.00133}=?$
- $\sqrt[3]{.000006859}=?$
- $\sqrt[3]{.002198}$

Ans. .3317; .267; 3.6.

- 4.
- $\sqrt[3]{6451699.37}=?$
- $\sqrt[3]{32.8119}=?$
- $\sqrt[3]{1405.3212}=?$

Ans. 2540.0196; 3.2014; 11.201.

- 5.
- $\sqrt{\frac{3}{5}}=?$
- $\sqrt{\frac{5}{7}}=?$
- $\sqrt{\frac{7}{8}}=?$
- $\sqrt{\frac{8}{9}}=?$

- 6.
- $\sqrt[3]{25.32}=?$
- $\sqrt[3]{.005}=?$
- $\sqrt[3]{.0000007}=?$

248. Though the extracting of roots of numbers approximately has already been touched upon, a general review of the subject will not be out of place here. It is to be noted that the demonstrations of the principles concerning approximate roots having been perfectly general, the rules of Article 233-3* apply, *verbatim*, to the case of numbers. The following special rule is a direct consequence of those rules. To extract the n th root of an integer (that is an imperfect

power of the indicated degree), to any required number of decimal places:

RULE.—Annex zeroes until the whole number annexed is n times the number of decimal places required in the root. Extract the n th root of the resulting number to within 1, and point off in the root the required number of decimal places.

249. The binomial formula also furnishes a means of extracting the n th root of an imperfect n th power to any required degree of accuracy. As x and y in the formula may represent any quantities whatever, we may divide the given number into any two parts, and let x represent one part and y the other, and then make the requisite substitutions in the expansion of $(x+y)^{\frac{1}{n}}$, n being the index of the required root. In practice it is found most convenient to let x represent the perfect power of the n th degree nearest to the given number; then, as $x+y$ equals the given number, y represents the difference between this perfect power and the given number. Thus $35=36+(-1)$. Hence, to obtain the square root of 35 we should substitute for x , 36, for y , -1 , and for n , 2, in the development. To get the seventh root of 126, we may place $x=128$; then as $x+y=126$, y must be the difference between 126 and 128, or -2 ; the expression for the root would therefore be $(128-2)^{\frac{1}{7}}$. The operation here indicated is expressed in the following

RULE.—Substitute for x in the development of $(x+y)^{\frac{1}{n}}$, the perfect power of the degree indicated, that is nearest to the given number. Subtract this from the given number, and substitute the result for y ; substitute for n the index of the required root. Carry out the development to as many terms as may be desirable and perform the operations indicated.

EXAMPLES.

$$1. \sqrt[3]{6} = (8-2)^{\frac{1}{3}} = 2 - \frac{1}{6} - \frac{1}{72} - \frac{5}{2592} - \dots$$

$$2. \sqrt[5]{125} = ? \sqrt[3]{3} = ? \sqrt[4]{4} = ? \sqrt[6]{27} = ? \sqrt[5]{8} = ? \sqrt[4]{18} = ?$$

CHAPTER VI. RADICALS.

DEFINITIONS.

250. A Radical is the indicated root of an imperfect power of the degree indicated.

Thus, $\sqrt[3]{4}$, $\sqrt[3]{63}$, $\sqrt[3]{m+n}$, are radicals.

251. A Rational Quantity is a quantity in the expression of which no radical occurs.

252. When a quantity is preceded by the sign of Evolution, three cases may arise.

1°. *The indicated root may be extracted, i. e., the quantity to which the radical sign is prefixed, is a perfect power of the degree indicated. Such an indicated root of a perfect power of the degree indicated, is called a Rational Quantity in a Radical Form.*

Thus, $\sqrt[3]{25}$, $\sqrt[3]{729}$, $\sqrt[3]{a^m}$, $\sqrt[3]{a^3+3a^2b+3ab^2+b^3}$, are rational quantities in radical form.

2°. *The indicated root cannot be extracted. The given expression is then a true Radical.*

Thus, $\sqrt[5]{2}$, $\sqrt{a^2+3a^2b+3ab^2+b^3}$, $\sqrt[4]{a^5}$, $\sqrt[10]{3}$, are radicals.

3°. *The indicated operation may be partly, but not entirely, performed; i. e., we may be able to extract the indicated root of a part only of the given expression. Such expressions, partly rational, partly radical, are generally classed as Irrational expressions. They are sometimes called Surds.*

Thus, $a+\sqrt{b}$, $3-\sqrt[3]{7}$, $5+2\sqrt{21+7\sqrt[3]{3}+a}$, are irrational expressions being equivalent to $\sqrt{a^2}+\sqrt{b}$, etc.

253. A Polynomial Surd is a polynomial, every term of which is a radical, as $\sqrt{a+\sqrt[3]{a^2b}+\sqrt[10]{7}+\sqrt{5}}$.

254. Polynomial Surds are subdivided into *Binomial*, *Residual*, *Trinomial*, etc., surds, as they take the form of *Binomials*, *Residuals*, etc.

255. The Degree of a radical is the degree of the indicated root.

Thus, $\sqrt{a}$, $\sqrt[3]{7}$, $\sqrt{m+n}$, are of the second degree.
 $\sqrt[5]{c}$, $\sqrt[5]{a+b}$, $\sqrt[5]{5m+2x}$, are of the fifth degree.
 $\sqrt[m]{2}$, $\sqrt[m]{x}$, $\sqrt[m]{a+b}$, are of the m th degree.

256. The Index of a radical is the index of the root required. When no index is written, the index 2 is understood.

The fractional exponents by which, as already stated, evolution may be indicated, may be considered as resulting from the application of the rules for evolution. Thus, to obtain $\sqrt[4]{a}$ we divide the exponent (1) of a by 4; to obtain the n th root we divide the exponent by n , and so on, whence,
 $\sqrt[4]{a} = a^{\frac{1}{4}}$, $\sqrt[n]{a} = a^{\frac{1}{n}}$, etc.

257. The Coefficient of a radical is a symbol placed before the radical to indicate the number of times it is to be taken. If no coefficient be written, 1 is understood.

258. A radical is in its Simplest Form when the quantity under the radical sign is integral, and contains no factor that is a perfect power of the degree indicated.

Thus, $\sqrt[3]{3}$ is in its simplest form; so, also, $\sqrt[3]{5}$, $\sqrt{a}$, $\sqrt[3]{a+b}$.
 But $\sqrt[3]{8} = 4 \times 2$, $\sqrt[3]{a^3b + a^{n+1}} = a^n(b+a)$, $\sqrt{\frac{1}{b}} = \frac{1}{\sqrt{b}} = \frac{1}{b} \times \left(\frac{1}{b}\right)^{\frac{1}{2}}$,
 are not in their simplest form.

259. The Transformation of a radical is the process of changing its form without altering its value.

260. Similar Radicals are such, as, being in their simplest form, have the same index, and the same quantity under the radical sign. Radicals not conforming to these conditions are said to be **Dissimilar**.

Thus, $\sqrt{a}$, $2\sqrt{a}$, $3\sqrt{a}$, $c\sqrt{a}$, are similar.

$\sqrt[3]{a}$, $2\sqrt[3]{a}$, $3\sqrt[3]{a}$, $d\sqrt[3]{a}$, are similar.

$\sqrt[3]{a}$, $\sqrt{a}$, are dissimilar; so, also, $\sqrt{a+b}$, $\sqrt{c+d}$.

It will be seen that, using the word coefficient in its proper or most general sense, *Similar Radicals* are *Similar Terms*; thus, $\sqrt{abc}$, $3\sqrt{abc}$, $7\sqrt{abc}$, may be written,

$$a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}, 3a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}, 7a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}},$$

so that we have, in these terms, the *same letters* with the *same exponents*. Again, take $a\sqrt{x}$, $b\sqrt{x}$, $c\sqrt{x}$. Here, a , b , and c , are the coefficients, and $\sqrt{x}=x^{\frac{1}{2}}$ is the *common unit* of the terms; $a\sqrt{xy}$, $b\sqrt{xy}$, $l\sqrt{xy}$, may be written,

$$ax^{\frac{1}{2}}y^{\frac{1}{2}}, bx^{\frac{1}{2}}y^{\frac{1}{2}}, lx^{\frac{1}{2}}y^{\frac{1}{2}},$$

so that the terms have the common unit $x^{\frac{1}{2}}y^{\frac{1}{2}}$, which is taken, in each case, the number of times indicated by the corresponding coefficient.

261. An Imaginary Quantity is the indicated *even* root of a negative quantity.

Thus, $\sqrt{-a}$, $\sqrt[4]{-x}$, $\sqrt[2m]{-xyz}$, $\sqrt[n]{-l}$, $\sqrt[4]{-a^4}$, are imaginary.

These quantities are called imaginary for the reason that they do not, cannot exist, since all even powers of positive quantities are positive, and all even powers of negative quantities are positive, as shown in (198). Hence there is no quantity which, raised to a power of an even degree, will produce a negative result, and a negative quantity can have therefore no root of an even degree.

A polynomial will be imaginary if any of its terms be imaginary.

262. A Real Quantity is a quantity that is not imaginary.

263. To Rationalize an irrational expression is to perform such operations upon the quantities therein, as will cause the radical signs to disappear.

SECTION I.

RADICALS OF THE SECOND DEGREE.

(4.) TRANSFORMATIONS.

284. I. To Deduce a Rule for reducing a Radical to its Simplest Form.

Let $x\sqrt{\frac{4ab^3}{c}}$ represent any radical of the second degree that is to be reduced to its simplest form.

Since the quantity under the radical sign in the result is to be integral, $\frac{4ab^3}{c}$ must be so transformed that the denominator shall be a perfect square; thus: $\frac{4ab^3c}{c^2}$. The radical is now $x\sqrt{\frac{4ab^3c}{c^2}}$, or, factoring the quantity under the sign, $x\sqrt{ac \cdot \frac{4b^3}{c^2}} = x\sqrt{\frac{4b^3}{c^2}}\sqrt{ac}$, since (217) the square root of a product equals the product of the square roots of the factors, $\sqrt{\frac{4b^3}{c^2}} = \frac{2b}{c}$. Hence $x\sqrt{\frac{4b^3}{c^2}}\sqrt{ac} = x\frac{2b}{c}\sqrt{ac} = \frac{2bx}{c}\sqrt{ac}$, which is in its simplest form. Had the denominator not been made a perfect square, c would not have been in the square root of the perfect square; it would therefore have been left under the sign, and a fraction would have occurred under the sign, after we had extracted the square root of the highest perfect square, so that the resulting radical would not have been in its simplest form.

If $c=1$, the quantity originally under the radical sign is itself integral.

As the letters here employed may represent any quantities whatever, we deduce the following

RULE.—*Transform the quantity under the radical sign so as to make the denominator a perfect square. Separate the result into two factors; one the highest perfect square in the quantity, the other containing no perfect square as a factor.*

Extract the square root of the first factor; multiply this result by the given coefficient, and annex the radical sign, under which place the second of the factors just mentioned.

EXAMPLES.

Reduce the following radicals to their simplest form:

$$\begin{aligned} 1. \quad 7\sqrt{605a^4b^3xyz^2} &= 7\sqrt{121a^4b^2z^2 \cdot 5xy} = 7 \times 11a^2b^2z \sqrt{5xy} \\ &= 77a^2b^2z \sqrt{5xy}. \end{aligned}$$

$$2. \sqrt{588a^5b^6c(x+y)^3}; \sqrt{512a^{-6}b^{-2n}c^2d^2}; \sqrt{484m^6b^7c^3}; \sqrt{448x^2y^2z^3}$$

$$3. \sqrt{\frac{768a^5(x+y)^{2n}c^3d^2}{784x^6(y+z)^{6rn}cd^2}} = \sqrt{\frac{768a^5(x+y)^{2n}c^3d^2n}{784x^6(y+z)^{6rn}c^3d^2}} = \\ \sqrt{\frac{256a^4(x+y)^{2n}c^3d^2}{784x^6(y+z)^{6rn}c^3d^2}} \sqrt{3acd n} = \sqrt{\frac{16a^4(x+y)^{2n}}{49x^6(y+z)^{6rn}}} \sqrt{3acd n} \\ = \frac{4a^2(x+y)^n}{7x^3(y+z)^{3rn}} \sqrt{3acd n}$$

$$4. \sqrt{\frac{3}{8}}; \sqrt{288}; \frac{1}{2}\sqrt{\frac{3}{7}}; \sqrt{\frac{ab^3cd}{4(a+x)}}; \sqrt{\frac{a(c-x)^2}{a-c+x}}; 3\sqrt{\frac{2}{5}}.$$

$$5. \sqrt{196a^3b^3c^3}; \sqrt{90a^5b^3(x+y)^3}; \sqrt{a^5-a^4x}; \sqrt{192a^7b^3(x+y)^7}.$$

$$6. x-y\sqrt{\frac{x+y}{x-y}}; \frac{a^5b^6}{c^3}\sqrt{\frac{c^7}{a^2b^3}}; 6\sqrt{3\frac{1}{2}}; 7\sqrt{8\frac{1}{3}}; 11\sqrt{9\frac{1}{6}}.$$

$$7. \frac{5}{2}\sqrt{\frac{1}{2}}; 5\sqrt{\frac{5}{18}}; 3\sqrt{\frac{2}{7}}; 12\sqrt{\frac{5}{8}}; 15\sqrt{\frac{7}{15}}; \sqrt{1183x^7y^2z^2w}.$$

$$Ans. \frac{5}{4}\sqrt{\frac{1}{2}}; \frac{5}{6}\sqrt{\frac{5}{10}}; \frac{3}{7}\sqrt{\frac{14}{14}}; 3\sqrt{\frac{10}{10}}; \sqrt{\frac{105}{105}}; 13x^2y^2z^2\sqrt{7xzw}.$$

8.

$$\sqrt{\frac{a^2-x^2}{a^2+x^2}}; \frac{a}{a^2-b^2}\sqrt{\frac{a^3b^2-2a^2b^3+ab^4}{b^5}}; \sqrt{(a^4-b^4)(a^2+b^2)}; \sqrt{832m^3x^6y}$$

$$Ans. \frac{1}{a^2+x^2}\sqrt{a^4-x^4}; \frac{a}{ab^3+b^3}\sqrt{ab}; (a^2+b^2)\sqrt{a^2-b^2}; 8mx^2\sqrt{13xy}.$$

$$9. \sqrt{27}; \sqrt{147x^2y^2c}; xy\sqrt{x^2y^2-x^2y^3}; (x+y)\sqrt{x^2-2x^2y+x^2y^2}; \\ 5\sqrt{60x^3y}.$$

$$10. \sqrt{216a^3x^2y}; \sqrt{x^2-x^3+x^4}; \frac{1}{x-y}\sqrt{x^2-xy^2-x^2y+y^2}; \\ \frac{x^3-y^3}{x+y}\sqrt{\frac{3x^2+6xy+3y^2}{7(x^2-y^2)y^2}}.$$

265. II.—To Deduce a Rule for introducing a factor under the radical sign.

Let the given expression be of the form $a\sqrt{x}$, and let it be required to introduce a under the sign.

Since $a=\sqrt{a}$ $\sqrt{a}=\sqrt{a^2}$, $a\sqrt{x}=\sqrt{a^2}\sqrt{x}=\sqrt{a^2x}$ (217).

As the letters here employed may represent any quantities whatever, we deduce the following

RULE.—*Square the quantity to be placed under the sign, and multiply this square into the quantity already under the sign.*

EXAMPLES.

$$\begin{array}{llllll} 7\sqrt{54}=? & 9\sqrt{57}=? & 16\sqrt{11}=? & 3\sqrt{52}=? & 7\sqrt{35}=? \\ 9\sqrt{56}=? & 10\sqrt{19}=? & 11\sqrt{31}=? & 5\sqrt{7}=? & 7\sqrt{5}=? \\ 7\sqrt{\frac{1}{7}}=? & 10\sqrt{\frac{1}{10}}=? & 6\sqrt{8}=? & 3\sqrt{7}=? & 2\sqrt{13}=? \end{array}$$

266. Scholium.—*This transformation is used to obtain the value of a given radical to within 1.*

Thus, let it be required to find $a\sqrt{b}$. By (211-8-9) we obtain $\sqrt{b}$ to within 1. When, however, we multiply this value by a , the product may differ from the true value of $\sqrt{b}$ by more than 1. But if we introduce a under the radical sign, we obtain $\sqrt{a^2b}$, an expression the value of which we may obtain at once (211-8-9) to within 1.

To ILLUSTRATE, let $a=10$, b , 13. We wish to find, then, the value of $10\sqrt{13}$. Now, $\sqrt{13}=3$ to within less than 1, and if we multiply this by 10, we obtain 30. 3 differs from the true root by more than $\frac{1}{2}$, hence 10 times 3 differs from the true root by more than 10 times $\frac{1}{2}$, or $\frac{10}{2}=5$. But $10\sqrt{13}=\sqrt{1300}=36$ to within 1. Hence the true value of this radical is 36, to within 1, and not 30; the error by taking 30 being even greater than that pointed out, viz. 5.

Similarly, $5\sqrt{2}=5\times 1=5$ by simply performing the operations as indicated. But $5\sqrt{2}=\sqrt{50}=7$, to within less than 1.

(B.) OPERATIONS ON RADICALS.

I. ADDITION.

267. To Deduce a Rule for adding Radicals.

I. Similar Radicals.

Since the coefficients, in each of the given quantities, show how often the common radical part is to be taken, additively or subtractively, the sum of the coefficients must show how

often this common radical part is to be taken in the aggregate of the given quantities. Hence the sum of the given quantities will consist of the common radical part with a coefficient equal to the algebraic sum of the given coefficients.

The addition of radicals may be simplified by the use of fractional exponents.

Thus, let it be required to add $a\sqrt{b}$ and $c\sqrt{b}$, a and c being positive or negative, entire or fractional,

$$a\sqrt{b} + c\sqrt{b} = ab^{\frac{1}{2}} + cb^{\frac{1}{2}} = (a+c)b^{\frac{1}{2}} = (a+c)\sqrt{b}.$$

From the foregoing considerations we deduce the following

RULE.—*To the algebraic sum of the coefficients annex the common radical part.*

268. Scholium I.—Since the letters employed in algebra may represent any quantities whatever, we may represent $\sqrt{b}$ in (267) by r , and then the addition of $a\sqrt{b}$ and $c\sqrt{b}$ falls under the ordinary rules of addition.

EXAMPLES.

1. Find the sum of $\sqrt{1620a^2b}$, $\sqrt{1445a^2b+170ab^2+5b^3}$, $\sqrt{80a^2b}$, $\sqrt{45a^2b-180ab^2+180b^3}$ and $\sqrt{45b^3+20a^2b+60ab^2}$.

Transforming the radicals, we obtain $18a\sqrt{5b}$, $4a\sqrt{5b}$, $(17a+b)\sqrt{5b}$, $(3a-6b)\sqrt{5b}$, and $(2a+3b)\sqrt{5b}$. The sum of the coefficients being $44a-2b$, the sum of the radicals is $(44a-2b)\sqrt{5b}$.

2. Add $\sqrt{150}$ and $-\sqrt{54}$; also, $\sqrt{4ax^3}$ and $3x\sqrt{9a}$.

Ans. $2\sqrt{6}$; $11x\sqrt{a}$.

3. Add $\sqrt{\frac{a^2x}{b^3}}$, $\sqrt{\frac{a^2x^3}{bc^2}}$ and $\sqrt{\frac{a^2c^2x}{bd^2}}$.

Ans. $\left(\frac{a^2}{b^3} + \frac{ax}{bc} + \frac{ac}{bd}\right)\sqrt{bx}$.

4. Add $\sqrt{24}$, $\sqrt{54}$ and $-\sqrt{6}$.

Ans. $4\sqrt{6}$.

5. Add $2\sqrt{8}$, $-7\sqrt{18}$, $5\sqrt{72}$ and $-\sqrt{50}$.

Ans. $8\sqrt{2}$.

6. Add $x\sqrt{3ay}$, $5y\sqrt{3ax}$, $2\sqrt{27axy^3}$ and $2y\sqrt{27ax^3}$.
7. Add $\sqrt{a^2x-a^2y}$, $\sqrt{36x-36y}$ and $\sqrt{9(a^2x-a^2y)}$.
8. Add $3\sqrt{3a^2x}$, $\sqrt{12a^2x}$ and $\sqrt{3b^2x}$. *Ans.* $(5a+b)\sqrt{3x}$.
9. Add $8\sqrt{\frac{2}{3}}$, $5\sqrt{\frac{2}{24}}$, $3\sqrt{\frac{2}{96}}$ and $5\sqrt{\frac{1}{6}}$.
10. Add $\sqrt{\frac{a^2x^2-2ax^4+x^6}{2a^2+4ax+2x^2}}$ and $\sqrt{\frac{a^2x^2+2ax^4+x^6}{2a^2-4ax+2x^2}}$.
Ans. $x\left(\frac{a^2+x^2}{a^2-x^2}\right)\sqrt{2x}$.
11. Add $2\sqrt{\frac{1}{2}}$, $\sqrt{\frac{2}{9}}$, $-\sqrt{18}$, $-\sqrt{\frac{25}{18}}$ and $\sqrt{\frac{9}{8}}$.
Ans. $-\frac{7}{4}\sqrt{2}$.
12. Add $.81\sqrt{5}$, $-\sqrt{1.8}$, $.4\sqrt{10}$, $6\sqrt{.05}$ and $-4\sqrt{0.2}$.
13. Add $\sqrt{\frac{x^2}{a}}$ and $\sqrt{\frac{a^2}{x}}$. *Ans.* $\frac{x^2+a^2}{ax}\sqrt{ax}$.
14. Add $8\sqrt{\frac{3}{4}}$, $\sqrt{60}$, $-\frac{11}{5}\sqrt{15}$ and $\sqrt{\frac{3}{5}}$. *Ans.* $4\sqrt{3}$.
15. Add $\sqrt{a^2bc^{-1}}$, $\sqrt{b^2ac^{-1}}$ and $\sqrt{abc}$. *Ans.* $\frac{a+b+c}{c}\sqrt{abc}$.

269. II.—Dissimilar Radicals.—Since we cannot add things unless they are of the same kind, *i. e.*, have the same unit, we cannot add dissimilar radicals, for they are things of different kinds. Hence, all that we can do is to indicate the addition. Hence, the

RULE.—Connect the radicals by their given signs.

270. Scholium I.—Always reduce the given radicals to their simplest form before attempting to decide whether they are similar or not.

271. Scholium II.—We may often be able to simplify slightly the result obtained by the rule. Thus $a\sqrt{bc} + m\sqrt{xc} = a\sqrt{b}/\sqrt{c} + m\sqrt{x}/\sqrt{c} = (a\sqrt{b} + m\sqrt{x})/\sqrt{c}$. This simplification is possible whenever the quantities under the radical signs have a common factor.

272. Scholium III.—*Using fractional exponents, dissimilar radicals give terms entirely or partly dissimilar. In the first case no simplification is possible; in the second we may slightly simplify our result. Thus:*

$$\begin{aligned} a\sqrt{b}+c\sqrt{d}&=ab^{\frac{1}{2}}+cd^{\frac{1}{2}} \\ a\sqrt{bc}+d\sqrt{xc}&=ab^{\frac{1}{2}}c^{\frac{1}{2}}+dx^{\frac{1}{2}}c^{\frac{1}{2}}=(ab^{\frac{1}{2}}+dx^{\frac{1}{2}})c^{\frac{1}{2}} \end{aligned}$$

EXAMPLES.

1. Add $\sqrt{x^2y}+\sqrt{xy^2}$ Ans. $x\sqrt{y}+y\sqrt{x}$.
2. Add $\sqrt{a^2x^2b}$, $\sqrt{c^2x^2b^2}$, and $\sqrt{d^2b}$
3. Add $\sqrt{72a^2b}$, $\sqrt{128c^2b}$, and $\sqrt{8a^2b}$

II. SUBTRACTION.

273. To Deduce a Rule for subtracting one radical from another.

I. Similar Radicals.

Let $\pm a\sqrt{c}$ and $\pm b\sqrt{c}$ be the minuend and subtrahend respectively. Since subtraction is the process of finding a quantity which, added to the subtrahend, will produce the minuend, it follows, from (267-9), that

$$\begin{aligned} (1) \quad +a\sqrt{c}-(+b\sqrt{c})&=(a-b)\sqrt{c}, \text{ e. g. } 12\sqrt{c}-7\sqrt{c}=5\sqrt{c} \\ \text{for } (a-b)\sqrt{c}+b\sqrt{c}&=a\sqrt{c} \dots \quad 5\sqrt{c}+7\sqrt{c}=12\sqrt{c} \end{aligned}$$

$$\begin{aligned} (2) \quad +a\sqrt{c}-(-b\sqrt{c})&=(a+b)\sqrt{c}, \\ \text{e. g. } 12\sqrt{c}-(-7\sqrt{c})&=19\sqrt{c} \\ \text{for } (a+b)\sqrt{c}+(-b\sqrt{c})&=a\sqrt{c}, \quad 19\sqrt{c}-7\sqrt{c}=12\sqrt{c} \end{aligned}$$

$$\begin{aligned} (3) \quad -a\sqrt{c}-(+b\sqrt{c})&=(-a-b)\sqrt{c}, \\ \text{e. g. } -12\sqrt{c}-7\sqrt{c}&=-19\sqrt{c} \\ \text{for } (-a-b)\sqrt{c}+b\sqrt{c}&=-a\sqrt{c}, \quad -19\sqrt{c}+7\sqrt{c}=12\sqrt{c} \end{aligned}$$

$$\begin{aligned} (4) \quad -a\sqrt{c}-(-b\sqrt{c})&=(-a+b)\sqrt{c}, \\ \text{e. g. } -12\sqrt{c}-(-7\sqrt{c})&=-5\sqrt{c} \\ \text{for } (-a+b)\sqrt{c}-b\sqrt{c}&=-a\sqrt{c}, \quad -5\sqrt{c}-7\sqrt{c}=-12\sqrt{c} \end{aligned}$$

Since the letters here employed may represent any quantities whatever, we deduce the following

RULE.—*Change the sign of the subtrahend or conceive it to be changed, and proceed as in addition.*

Scholium.—*By placing $\sqrt{0=r}$, we could apply the discussion of (49-52) and resubstitute in the result. Again, by employing fractional exponents, we may reduce this case to (49-52).*

274. II. Dissimilar Radicals.

Since we cannot subtract a number of things from another number of things, unless all the things considered be of the same kind, we cannot subtract a radical from a dissimilar one, for these are things of different kinds; the coefficients indicate the number of times we are to take different things. Hence we can merely indicate the operation, simplifying only by combining the signs of operation and quantity, to obtain the essential sign (57).

Hence the

RULE.—*Change the sign of the subtrahend and annex the result to the minuend.*

Scholium.—The principles of (269) may be applied here.

EXAMPLES.

$$1. \sqrt{\frac{3}{5}} - \sqrt{\frac{5}{27}} = ? \quad \sqrt{\frac{3}{5}} = \frac{1}{5}\sqrt{15}; \quad \sqrt{\frac{5}{27}} = \frac{1}{9}\sqrt{15}.$$

$$\text{Hence } \sqrt{\frac{3}{5}} - \sqrt{\frac{5}{27}} = \left(\frac{1}{5} - \frac{1}{9}\right)\sqrt{15} = \left(\frac{9-5}{45}\right)\sqrt{15} = \frac{4}{45}\sqrt{15}$$

$$2. \frac{19}{30}\sqrt{6} - \sqrt{\frac{27}{50}} = ? \quad 10\sqrt{3} - \sqrt{27} = ? \quad 5\sqrt{6} - \sqrt{54} = ?$$

$$\sqrt{243} - \sqrt{75} = ? \quad \text{Ans. } \frac{1}{3}\sqrt{6}; 7\sqrt{3}; 2\sqrt{6}; 4\sqrt{3}.$$

$$3. \sqrt{2} - \sqrt{\frac{1}{2}} = ? \quad \sqrt{192} - \sqrt{75} = ? \quad \sqrt{45} - \sqrt{0.8} = ?$$

$$\sqrt{108} - \sqrt{363} = ? \quad \text{Ans. } \frac{1}{2}\sqrt{2}; 3\sqrt{3}; 2.6\sqrt{5}; -5\sqrt{3}.$$

4. $5\sqrt{567}-3\sqrt{1008}=?$ $9\sqrt{1620}-8\sqrt{3380}=?$
 $\sqrt{1682}-\sqrt{1458}=?$ *Ans.* $9\sqrt{7}$; $-23\sqrt{20}$; $2\sqrt{2}$.
5. $\sqrt{75a^2x+9b^2x}-\sqrt{12a^2x}-\sqrt{3b^2x}=?$ $\sqrt{a}-\sqrt{\frac{9a}{16}}=?$
 $\sqrt{100a^2b}-\sqrt{36a^2b}=?$ $\sqrt{5a^2b}-\sqrt{5ab^2}=?$
6. $\sqrt{2a^4+8a^2b+12a^2b^2+8a^2b+2b^4}-\sqrt{2a^4}-\sqrt{2b^4}=?$
 $\sqrt{12a^2b}+\sqrt{98x^2y}+\sqrt{75a^2b}-\sqrt{8x^2y}+\sqrt{36a^2b}-\sqrt{288y}=?$

III. MULTIPLICATION.

275. To Deduce a Rule for multiplying Radicals.

Let $a\sqrt{x}$ be the multiplicand and $b\sqrt{y}$ the multiplier. The product of $a\sqrt{x}$ by $b\sqrt{y}$ may be indicated thus:

$$a\sqrt{x} \times b\sqrt{y}; \text{ or, (61) } ab\sqrt{x\sqrt{y}}; \sqrt{x\sqrt{y}}=\sqrt{xy} \text{ (217).}$$

Hence $a\sqrt{x} \times b\sqrt{y}=ab\sqrt{xy}$. Here we see that the coefficient of the product is the product of the coefficients, and the quantity under the radical sign of the product is the product of the corresponding quantities in the factors. Since the letters employed may represent any quantities whatever, we deduce, for multiplying radicals, the following

RULE.—*Multiply the coefficient of the multiplicand by the coefficient of the multiplier for the coefficient of the product. After this write the radical sign, under which place the product of the quantities under the radical signs in the given factors. Reduce the result to its simplest form.*

276. Scholium I.—If we are to take the product of several factors we multiply the product of the first two by the third, this result by the fourth, and so on, that is:

To the continued product of the coefficients, we annex the radical sign, under which we place the continued product of the quantities under the radical signs in the factors.

277. Scholium II.—*With fractional exponents the same result may be reached as in (275). Thus: $a\sqrt{x} \times b\sqrt{y}=ax^{\frac{1}{2}}by^{\frac{1}{2}}=abx^{\frac{1}{2}}y^{\frac{1}{2}}=ab(xy)^{\frac{1}{2}}=ab\sqrt{xy}$.*

EXAMPLES.

1. Multiply $2\sqrt{27}$ by $\sqrt{3}$; $\sqrt{2}$ by $\sqrt{5}$; $\sqrt{7}$ by $\sqrt{14}$; $\sqrt{9}$ by $\sqrt{54}$. Ans. 18 ; $\sqrt{10}$; $7\sqrt{2}$; $9\sqrt{6}$.

2. Multiply $\sqrt{.3} - \sqrt{.2} - \sqrt{.9}$ by $3\sqrt{.4} - 5\sqrt{.2} + 4\sqrt{.3} - 3\sqrt{.6}$. Ans. 0.4 .

3. Multiply $2 + \sqrt{2} + 2\sqrt{\frac{1}{3}}$

by $3\sqrt{\frac{1}{2}} - \sqrt{6} + \sqrt{3}$

$$\begin{array}{r} 6\sqrt{\frac{1}{2}} + 3 + 6\sqrt{\frac{1}{6}} - \sqrt{12} \\ - 2\sqrt{2} + 2 - 2\sqrt{6} + 2\sqrt{3} \\ + \sqrt{6} \end{array}$$

$$\frac{6}{2}\sqrt{2} - 2\sqrt{2} + 5 + \sqrt{6} - 2\sqrt{6} + \sqrt{6} - 2\sqrt{3} + 2\sqrt{3} = 5 + \sqrt{2}.$$

4. $(2 + 3\sqrt{2} + \sqrt{3})(6 - 9\sqrt{2} + 3\sqrt{3}) = ?$

$(\sqrt{10} - 2\sqrt{5} + \sqrt{30})(\sqrt{.4} - \sqrt{.2} + \sqrt{.6}) = ?$

Ans. $12\sqrt{3} - 33$; 4 .

5. Multiply $(2\sqrt{6} - 3\sqrt{5})$ by $(4\sqrt{3} - \sqrt{10})$

$(\sqrt{2} + \sqrt{3})(\sqrt{8} - \sqrt{12}) = ?$

6. Multiply $(5\sqrt{7} + 3\sqrt{8})$ by $(2\sqrt{\frac{1}{7}} - 4\sqrt{3} + 7\sqrt{\frac{1}{8}} - 2)$

7. $(8 - \sqrt{23})(8 + \sqrt{23}) = ?$ $(7 - \sqrt{15})(7 - \sqrt{15}) = ?$

8. $(9 - 3\sqrt{5})(9 - 8\sqrt{45}) = ?$ Ans. $441 - 243\sqrt{5}$.

9. $(3a + 5\sqrt{112ab^3x})(7b - 8\sqrt{252ab^3x}) = ?$

10. $(\sqrt{a} + \sqrt{b} + \sqrt{c})(\sqrt{a} + \sqrt{b} - \sqrt{c}) = ?$

$(8\sqrt{5} - 4\sqrt{3})(9\sqrt{\frac{1}{5}} - 2\sqrt{3} + 6\sqrt{7}) = ?$

11. Multiply $\frac{a+b}{(a-b)(a+1)}\sqrt{a^2-1}$ by $(a-b)\sqrt{a^2+2a^3+a}$.

Ans. $(a+b)\sqrt{a^2-a}$.

12. Multiply $\sqrt{a^7 - a^3} + \sqrt{a^5 - a^3} + \sqrt{a^3 - a} + \sqrt{a - 1}$ by $\sqrt{a + 1}$.
Ans. $a^4 - 1$.

13. Multiply $\sqrt{2a} + \sqrt{3b} + \sqrt{c}$ by $2\sqrt{a} - \sqrt{b} - \sqrt{3c^2}$.

14. Multiply $a^{3m} + \sqrt{a^{2m}b^{2m}} - b^{2m}$ by $\sqrt{a^{2m}} - a^{2m}\sqrt{b^{2m}} + b^{2m}\sqrt{a^{2m}} - \sqrt{b^{2m}}$.

Ans. $\sqrt{a^{12m}} - b^{2m}\sqrt{a^{2m}} + a^{2m}\sqrt{b^{2m}} - 2\sqrt{a^{2m}b^{12m}} + \sqrt{b^{12m}}$.

IV. DIVISION.

278. To Deduce a Rule for the division of Radicals.

Let it be required to divide $a\sqrt{x}$ by $b\sqrt{y}$. This operation may be indicated thus:

$$\frac{a\sqrt{x}}{b\sqrt{y}} = \frac{a}{b} \times \frac{\sqrt{x}}{\sqrt{y}} = \frac{a}{b} \sqrt{\frac{x}{y}} \quad (217).$$

The coefficient of this quotient equals the coefficient of the dividend divided by the coefficient of the divisor. The radical part of the quotient equals the square root of the quotient obtained by dividing the quantity under the radical sign in the dividend by the corresponding quantity in the divisor. As the letters employed may represent any quantities whatever, we deduce the following

RULE.—Divide the coefficient of the dividend by the coefficient of the divisor. To this quotient annex the radical sign, under which place the quotient obtained by dividing the quantity under the radical sign in the dividend by the corresponding quantity in the divisor.

Reduce the result to its simplest form.

279. Scholium.—The theory of fractional exponents leads to the same result as the application of the rule. Thus:

$$a\sqrt{x} \div b\sqrt{y} = ax^{\frac{1}{2}} \div by^{\frac{1}{2}} = \frac{ax^{\frac{1}{2}}}{by^{\frac{1}{2}}} = \frac{a}{b} \left(\frac{x}{y}\right)^{\frac{1}{2}} = \frac{a}{b} \sqrt{\frac{x}{y}}$$

EXAMPLES.

1. $\sqrt{\frac{(a^2+b^2)x}{(a^2-b^2)^2y}} \div \sqrt{\frac{1}{(a^2-b^2)^2x}} = ?$ *Ans.* $\frac{(a^2+b^2)x}{y} \sqrt{y}$
2. $30\sqrt{10} \div 5\sqrt{8} = ?$ $\sqrt{a^2-x^2} \div (a+x) = ?$
 $\frac{1}{10}\sqrt{6} \div \frac{2}{15}\sqrt{9} = ?$ *Ans.* $3\sqrt{5}$; $\sqrt{\frac{a-x}{a+x}}$; $\frac{1}{4}\sqrt{6}$.
3. $\frac{\sqrt{ab^2x-b^2cx}}{\sqrt{a-c}} = ?$ $a\sqrt{\frac{bc-bx}{c}} \div b\sqrt{\frac{ad-ax}{d}} = ?$
 $\sqrt{ax^3} \div \sqrt{bx} = ?$
Ans. $b\sqrt{x}$; $\sqrt{\frac{ad(c-x)}{bc(d-x)}}$; $\frac{x}{b}\sqrt{ab}$.
4. Divide $a\sqrt{x}-\sqrt{bx}+a\sqrt{y}-\sqrt{by}$ by $\sqrt{x}+\sqrt{y}$.
Ans. $a-\sqrt{b}$.
5. Divide $3\sqrt{15}-\sqrt{20}+\sqrt{10}-7$ by $2\sqrt{5}$.
Ans. $\frac{3}{2}\sqrt{3}-1+\frac{1}{2}\sqrt{2}-\frac{1}{10}\sqrt{245}$.
6. Divide $\sqrt{x^5}-x^3-4\sqrt{x^3}+6x-2\sqrt{x}$ by $\sqrt{x^3}-4\sqrt{x}+2$.
7. Divide $n^3+(n-1)x+(n-1)\sqrt{x^3}+(n-1)x^2-\sqrt{x^5}$
by $n-\sqrt{x}$. *Ans.* $n+\sqrt{x}+x+\sqrt{x^3}+x^2$.
8. Divide $-12b\sqrt{a^3}-20a\sqrt{b^3}+4a^2\sqrt{b}$ by $a-3\sqrt{ab}-5b$.
9. Divide $\sqrt{x^3}-\sqrt{x^2y}+\sqrt{xy^2}-\sqrt{y^3}$ by $\sqrt{x}-\sqrt{y}$.

V. INVOLUTION.

280. To Deduce a Rule for raising a Radical to any power.

Let it be required to raise $a\sqrt{x}$ to the n th power. This means that $a\sqrt{x}$ is to be taken n times as a factor, thus:—
 $a\sqrt{x}.a\sqrt{x}.a\sqrt{x}.a\sqrt{x} \dots$ to n factors. By (275) we should take the continued product of the coefficients for a new coefficient and to this annex the radical sign, placing under it the continued product of all the quantities under

the radical signs of the given factors. Now as $a\sqrt{x}$ occurs n times as a factor, a occurs n times and $\sqrt{x}$ occurs n times. The continued product of the coefficients is, then, the continued product of a ($aa \dots$) taken n times as a factor and is therefore a^n . The continued product of the radical parts is $\sqrt{xxxxxxxxx \dots}$ to n factors, i. e., x^n (15). Hence the whole product is $a^n\sqrt{x^n}$. A comparison of this result with the given quantity shows that the power might have been obtained by raising the coefficient to the required power, and after this writing the radical sign, raising the quantity under it to the required power. The letters employed representing any quantities whatever, we deduce the following

RULE.—*Raise the coefficient to the required power. After this place the radical sign, the quantity under it having been raised to the required power.*

Reduce the result to its simplest form.

281. Corollary.—*If the exponent of the required power be even, to raise the given quantity to this power we have the*

RULE.—*Raise the coefficient to the required power. After this place the quantity originally under the radical sign, the quantity being raised to a power the exponent of which is one-half the exponent of the required power.*

For, by the rule (280) $(a\sqrt{x})^n = a^n\sqrt{x^n}$. If, now, n be even, let it be equal to $2s$, then $(a\sqrt{x})^{2s} = a^{2s}\sqrt{x^{2s}}$. By (211) $\sqrt{x^{2s}} = x^s$. Hence $(a\sqrt{x})^{2s} = a^{2s}x^s$.

282. Scholium.—*Using fractional exponents we have*
 $(a\sqrt[n]{x})^n = (ax^{\frac{1}{n}})^n = (195), a^n x^{\frac{n}{n}}$. If, now, n be entire and even $x^{\frac{n}{n}}$ is rational; if n be odd $x^{\frac{n}{n}}$ is irrational, and the result, $a^n x^{\frac{n}{n}}$, is the same $(a^n\sqrt[n]{x^n})$ as that previously obtained.

EXAMPLES.

$$\begin{array}{lll}
 (3x\sqrt{5y})^7=? & (5x\sqrt{3y^2})^4=? & (7a^2by\sqrt{(a-x)b})^5=? \\
 (5a-7\sqrt{b})^3=? & (7x+3\sqrt{x^2y})^4=? & (3\sqrt{5x}-4\sqrt{3y})^6=?
 \end{array}$$

VI. EVOLUTION.

283. To Deduce a Rule for extracting any root of a Radical.

Evolution being the converse of Involution, to extract any root of a radical is to find a quantity which, raised to the power of which the exponent equals the index of the required root, will produce the given quantity. By (280) the coefficient of the quantity sought, must, on being raised to the power just mentioned, produce the coefficient of the given quantity. The quantity under the radical sign (in the root sought), must also, on being raised to the power mentioned, produce the quantity given under the radical sign. Hence the coefficient of any root of a radical will be the required root of the given coefficient. The quantity under the radical sign in the root will be the required root of the quantity given under the radical sign.

$$\text{Thus } \sqrt[n]{a^n \sqrt{x^n}} = a \sqrt{x}.$$

From the foregoing considerations we deduce the

RULE.—*Extract the required root of the given coefficient. After this place the radical sign, under which write the required root of the quantity given under the radical sign.*

Scholium.—It will be seen from the reasoning of 195, 213, 280 and 283, that the applicability of the rules of 280-3 is not affected by the form of the exponents or indices of the required powers or roots.

EXAMPLES.

Extract the square root of $\sqrt{ax^2}$; of $x+2\sqrt{xy}+y$; of $16\sqrt{x}+32\sqrt{x^{\frac{1}{2}}y^{\frac{1}{2}}}+16\sqrt{y}$; of $\sqrt{m^2n}$; of $\sqrt{a^2x-ax^2}$; of $784\sqrt{4ax^2y}$.

2. Extract the cube root of $103173\sqrt{21}$; or $(5a^2-3x^2y^2;\sqrt{5a^2-3x^2y^2};$ of $x\sqrt{x}+27y\sqrt{x}+9x\sqrt{y}+27y\sqrt{y}$; of $(27a^3-54ax+27x^2)\sqrt{a-x}$; of $\sqrt{a^2-3a}\sqrt{b}+3b\sqrt{a}-\sqrt{b^3}$.

3. Extract the fourth root of $625a\sqrt{a^3b^2}$; of $(x^2+3x^2y+3xy^2+y^2)\sqrt{(x+y)^6a^{12}b^8c}$; of $x^2-4\sqrt{x^2y}+6xy-4\sqrt{xy^2}+y^2$.

284. **Scholium.**—Using fractional exponents we have,

$$\sqrt[n]{a^n\sqrt{x^n}} = (a^n x^{\frac{n}{2}})^{\frac{1}{n}} = ax^{\frac{1}{2}} \text{ or } a\sqrt{x}.$$

IMAGINARY QUANTITIES OF THE SECOND DEGREE.

MONOMIALS.

285. **Lemma.**—Every imaginary quantity of the second degree may be reduced to the form $c\sqrt{-1}$, c being rational or irrational, but real.

Let $\sqrt{-p}$ be any imaginary quantity of the second degree. Place $\sqrt{-p}=r$. Then $\sqrt{-1}\times p=r$. Squaring both of these equals we have $-1\times p=r^2$.

Since we extract the square root of a product by dividing the exponent of each factor by 2, i. e., by extracting the square root of each factor, if we extract the square root of both of these equals we have $\sqrt{-1}\sqrt{p}=r=\sqrt{-p}$. Placing $\sqrt{p}=c$, we have $\sqrt{-p}=c\sqrt{-1}$.

286. **Corollary I.**—The product of the square root of a positive quantity and the square root of a negative quantity equals the square root of the product of the quantities. Thus:

$$\sqrt{a}.\sqrt{-b}=\sqrt{-ab} \text{ for } \sqrt{a}.\sqrt{-b}=\sqrt{a}\sqrt{b}\sqrt{-1}=\sqrt{-ab}$$

287. **Corollary II.**—The square root of a quantity divided by the square root of a quantity with a contrary sign equals the square root of the quotient of the first quantity by the second.

$$\text{Thus: } \sqrt{a}\div\sqrt{-x}=\sqrt{-\frac{a}{x}}; \sqrt{-x}\div\sqrt{a}=\sqrt{-\frac{x}{a}}$$

288. Corollary III.—*The rules 284–285 for transformation may be applied, without change, to imaginary quantities.*

I. ADDITION.

289. To Deduce Rules for adding imaginary quantities of the second degree.

I. Similar Radicals.

From the nature of coefficients and the reasoning in (267) it follows that if several terms have a common radical part, even if this part be imaginary, the algebraic sum of the coefficients will show how often the radical part is to be taken (added or subtracted, as the case may be) in the aggregate or sum of these terms. Hence to add imaginary quantities of the second degree we apply the rule, **Art. 267**, without change.

II. Dissimilar Radicals.—From the reasoning before given it is clear that the addition of these radicals can only be indicated, by connecting the simplified radicals by their proper signs.

II. SUBTRACTION.

290. To Deduce a Rule for the subtraction of imaginary quantities of the second degree.

From the nature of coefficients and the reasoning of (273–4) it follows that the rule there deduced may be applied, without modification, to imaginary quantities.

EXAMPLES.

$$1. \text{ Add } 3\sqrt{-\frac{2}{5}} \text{ and } 5\sqrt{-\frac{5}{18}}. \quad \text{Ans. } \frac{43}{30}\sqrt{-10}.$$

$$2. \text{ Add } 6\sqrt{-3\frac{1}{2}} \text{ and } 3\sqrt{-\frac{2}{7}}. \quad \text{Ans. } 3\frac{3}{7}\sqrt{-14}.$$

$$3. \text{ Add } \sqrt{-72}, \sqrt{-128} \text{ and } \sqrt{-162}. \quad \text{Ans. } 23\sqrt{-2}.$$

$$4. \text{ Add } \sqrt{-180}, \sqrt{-605}, \sqrt{-3920} \text{ and } \sqrt{-500}.$$

5. Add $2\sqrt{-\frac{5}{3}}$, $\sqrt{-60}$, $\sqrt{-15}$ and $\sqrt{-\frac{3}{5}}$.

Ans. $\frac{58}{15}\sqrt{-15}$.

6. Add $a\sqrt{-\frac{a}{bc}}$, $b\sqrt{-\frac{b}{ac}}$ and $c\sqrt{-\frac{c}{ab}}$.

Ans. $\frac{a^2+b^2+c^2}{abc}\sqrt{-abc}$.

7. Add $\frac{a}{b}\sqrt{-\frac{1}{c}}$, $\frac{b}{a}\sqrt{-\frac{1}{c}}$ and $\frac{c}{a}\sqrt{-\frac{1}{b}}$.

Ans. $\frac{a^2+b^2+\sqrt{bc^3}}{abc}\sqrt{-c}$.

8. Add $\frac{1}{2}\sqrt{-\frac{1}{3}}$, $\sqrt{-\frac{1}{6}}$ and $-\frac{5}{3}\sqrt{-\frac{1}{2}}$.

Ans. $\frac{\sqrt{3}+\sqrt{6}-5\sqrt{2}}{6}\sqrt{-1}$.

9. From $9a\sqrt{-bx^3}$ subtract $5x\sqrt{-ba^3}$. Ans. $4ax\sqrt{-b}$.

10. From $\sqrt{\frac{a^2x+2ax^2+x^3}{2ax-x^3-a^2}}$ subtract $\sqrt{\frac{2ax^2-x^3-a^2x}{x^3+2ax+a^2}}$.

Ans. $\frac{4ax}{a^2-x}\sqrt{-x}$.

11. From $\sqrt{x^3-4x^2-x-5x^3}+\sqrt{x^3-4xy^3-4x^2y-2x^3}$ subtract $\sqrt{6xy^3-4xy^2-10xy^2-x}+\sqrt{-4x^3}$. Ans. $x\sqrt{-x}$.

12. From $\sqrt{-18a^5b^3-60a^4b^4-50a^3b^5}$ subtract $\sqrt{-50a^5b^3}$.

Ans. $3a^2b\sqrt{-2ab}$.

III. MULTIPLICATION.

291. The rule deduced in Article (275) depends on (217), in which the quantities treated are assumed real. It requires some modification when the quantities treated are imaginary. Thus $\sqrt{-a} \cdot \sqrt{-a}$ can be only $-a$, from the definition of a root; while by the rule (275) $\sqrt{-a} \cdot \sqrt{-a} = \sqrt{a^2} = \pm a$. Again, by the rule $\sqrt{-a} \times \sqrt{-x} = \pm \sqrt{ax}$, whereas it must be $-\sqrt{ax}$, for, $\sqrt{-a} = \sqrt{a}\sqrt{-1}$; $\sqrt{-x} = \sqrt{x}\sqrt{-1}$. Hence the product

$$\sqrt{-a} \cdot \sqrt{-x} = \sqrt{a}\sqrt{x}\sqrt{-1}\sqrt{-1} = \sqrt{ax}(\sqrt{-1})^2.$$

From the definition of a root $(\sqrt{-1})^2 = -1$.

Hence $\sqrt{ax}(\sqrt{-1})^2 = \sqrt{ax}(-1) = -\sqrt{ax}$, or, if $\sqrt{ax}$ be ordinarily taken with the double sign $\pm$, this product will be $\mp \sqrt{ax}$.

292. To Deduce a Rule for multiplying together imaginaries of the second degree.

Let it be required to multiply $\sqrt{-x}$ by $\sqrt{-y}$. Transforming these radicals, as just indicated, we have,

$$\sqrt{x}\sqrt{-1} \cdot \sqrt{y}\sqrt{-1} = \sqrt{xy}(\sqrt{-1})^2 = -\sqrt{xy}.$$

If we have several factors, as $\sqrt{-a}$, $\sqrt{-b}$, $\sqrt{-c}$, $\sqrt{-d}$, $\sqrt{-x}$, we may evidently transform each of them, as just indicated, thus $\sqrt{a}\sqrt{-1}$, $\sqrt{b}\sqrt{-1}$, $\sqrt{c}\sqrt{-1}$, $\sqrt{d}\sqrt{-1}$, and so on. The product would then be the product of the real factors $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$, etc., into that power of $\sqrt{-1}$ of which the exponent is the number of imaginary factors. Hence the following

RULE.—Place each of the imaginary terms under the form $\sqrt{x}\sqrt{-1}$. Multiply the real factors, by the ordinary rules, and after the product place, as a multiplier, that power of $\sqrt{-1}$ of which the exponent equals the number of imaginary factors.

Reduce the result to its simplest form.

EXAMPLES.

1. $(a + \sqrt{b}\sqrt{-1})(a - \sqrt{-b}) = ?$ Ans. $a^2 + b$.
 $(2\sqrt{3} - \sqrt{-5})(4\sqrt{3} - 2\sqrt{-5}) = ?$ Ans. $14 - 8\sqrt{-15}$.
 $(6a + 4\sqrt{-b} + \sqrt{-c})(2a - a\sqrt{-b} + b\sqrt{-c}) = ?$
2. $(\sqrt{-15} + \sqrt{-17})(\sqrt{-30} - \sqrt{-115}) = ?$
 $(\sqrt{-a} + \sqrt{-ab} + \sqrt{-b})(\sqrt{a} + \sqrt{-b}) = ?$
3. $(5 + 8\sqrt{-7})(6 + 3\sqrt{-10}) = ?$
 $(x^2 + \sqrt{-y^3})(x^2 - \sqrt{-y^3}) = ?$
Ans. $30 + 48\sqrt{-7} + 15\sqrt{-10} - 24\sqrt{70}; x^4 + y^3$.
4. $(x + \sqrt{-y-1})(x - \sqrt{-y-1}) = ?$
 $(x - \sqrt{-(x+a)^2})(x + \sqrt{-(x+a)^2}) = ?$ Ans. $x^2 + (x+a)^2$.

293. The following table of the powers of $\sqrt{-1}$ will be found useful in applying the rule (292).

TABLE.

$(\sqrt{-1})^1 = \sqrt{-1}$	$= \sqrt{-1};$	$(\sqrt{-1})^{4n+1} = \sqrt{-1}$
$(\sqrt{-1})^2 = -1$	$= -1;$	$(\sqrt{-1})^{4n+2} = -1$
$(\sqrt{-1})^3 = (\sqrt{-1})^2 \times \sqrt{-1} = -\sqrt{-1}$	$= -\sqrt{-1};$	$(\sqrt{-1})^{4n+3} = -\sqrt{-1}$
$(\sqrt{-1})^4 = (\sqrt{-1})^2 \times (\sqrt{-1})^2$ $= -1 \times -1 = +1;$	$= +1;$	$(\sqrt{-1})^{4n+4} = +1$

The formation of the first four powers is evident. Since any power of $+1$ is $+1$, if n be the exponent of any power whatever, $1^n = 1$. We see that $(\sqrt{-1})^4 = 1$. Hence $(\sqrt{-1})^{4n} = 1^n = 1$. (For, if any two quantities are equal, the n th powers of them are equal; $(\sqrt{-1})^{4n}$ is the n th power of $(\sqrt{-1})^4$, and 1 is the n th power of 1). Since $(\sqrt{-1})^{4n} = 1$, no matter what power of $(\sqrt{-1})$ is multiplied by $(\sqrt{-1})^{4n}$ the product will be the same as the multiplier.

Hence, $(\sqrt{-1})^{4n} \times (\sqrt{-1})^1 = \sqrt{-1} = (\sqrt{-1})^{4n+1}$.
Similarly, $(\sqrt{-1})^{4n+2} = (\sqrt{-1})^{4n} \times (\sqrt{-1})^2 = 1 \times -1 = -1$,
 $(\sqrt{-1})^{4n+3} = (\sqrt{-1})^{4n} \times (\sqrt{-1})^3 = 1 \times -\sqrt{-1} = -\sqrt{-1}$,
and $(\sqrt{-1})^{4n+4} = (\sqrt{-1})^{4n} \times (\sqrt{-1})^4 = 1 \times 1 = +1$.

Hence the powers of $\sqrt{-1}$ indicated will be as found in the table. By making $n=0, 1, 2, 3, 4, \dots, 4n+1, 4n+2, 4n+3, 4n+4$, become $1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, \dots$ so that we may find *any power* of $\sqrt{-1}$ from the table by substituting for $4n$, the greatest multiple of 4 in the exponent of the power.

294. Scholium I—It should be noted that $-\sqrt{xy}$, the product of $\sqrt{-x}$ and $\sqrt{-y}$, by the foregoing rule, signifies that $\sqrt{xy}$ is to be taken in a sense opposite to that in which it is ordinarily taken. Though the sign $\pm$ is not generally *written* before a radical, it is *always understood*, and the product $-\sqrt{xy}$ means $-(\pm\sqrt{xy}) = \mp\sqrt{xy}$, and does not mean that the minus sign of the root is the *only* one that

can be taken. The minus sign indicates, in other words, that if in the expression $\sqrt{xy}$ the *positive* value is to be taken *first*, in the expression $-\sqrt{xy}$ the *negative* value is to be taken *first*.

295. Scholium II.—It will also be noted that if the number of imaginary factors is even, the product is real; if this number is odd, the product is imaginary. When the product is real, it is positive if the number of factors is a multiple of 4, and negative if this number is not a multiple of 4. If, the product being imaginary, the number of factors divided by 4 leaves a remainder of 1, it is positive, otherwise it is negative.

IV. DIVISION.

296. To Deduce a Rule for the division of Imaginaries.

The rule of **Art. (278)** depends on the lemma (217), which is equally true if the two quantities are imaginary.

$$\text{For } \frac{\sqrt{-x}}{\sqrt{-y}} = \frac{\sqrt{x}\sqrt{-1}}{\sqrt{y}\sqrt{-1}} = \frac{\sqrt{x}}{\sqrt{y}} = \sqrt{\frac{x}{y}} = \sqrt{\frac{-x}{-y}}.$$

If one only of the quantities be imaginary we have (287),

$$\sqrt{x} \div \sqrt{-y} = \sqrt{-\frac{x}{y}}.$$

Hence the rule (278) is equally applicable whether the quantities be real or imaginary.

EXAMPLES.

$$1. 25\sqrt{-ab^3} + 5\sqrt{-b} = ? \quad 372(a-x)\sqrt{a-x} \div 3\sqrt{(a-x)^5} = ?$$

$$2. (\sqrt{a} + 64\sqrt{b}) \div \sqrt{a^{\frac{1}{3}}} + 4\sqrt{b^{\frac{1}{3}}} = ?$$

$$\text{Ans. } a^{\frac{1}{3}} - 4\sqrt{a^{\frac{1}{3}}b^{\frac{1}{3}}} + 16b^{\frac{1}{3}}.$$

$$3. (a^4 - 1) \div (\sqrt{-a} + 1) = ?$$

$$\text{Ans. } -\sqrt{-a^3} + a^3 + \sqrt{-a^5} - a^2 - \sqrt{-a^7} + a + \sqrt{-a} - 1.$$

$$4. (a^2x + a\sqrt{-xy} - ax\sqrt{-z} + c\sqrt{xyz}) \div (a - \sqrt{-z}) = ?$$

$$5. 1 \div \frac{a - \sqrt{-x}}{a + \sqrt{-x}} = ? \quad \frac{a + \sqrt{-b}}{a - \sqrt{-b}} \div \frac{a - \sqrt{-b}}{a + \sqrt{-b}} = ?$$

V. INVOLUTION.

297. To Deduce a Rule for raising an imaginary quantity of the second degree to any power.

From the reasoning of (292) we have the following

RULE.—Place the given radical under the form $\sqrt{x}\sqrt{-1}$. Raise the real factor to the required power, and multiply this by the required power of $\sqrt{-1}$, as found in the table (293).

298. Scholium.—If the exponent of the power be even, the power will be real; if this exponent be odd, the power will be imaginary. In the first case the power will be positive or negative, as the exponent is divisible by 4, or not. In the second case it will be positive or negative according as the remainder left after dividing the exponent by 4 is 1 or 3.

$$\begin{aligned}\text{Ex. } (3\sqrt{-1})^{10} &= ? & (a + \sqrt{-b})^4 &= ? & (7\sqrt{-a+x})^7 &= ? \\ (4\sqrt{-a^2})^6 &= ? & (x \pm \sqrt{-y})^6 &= ? & (xy \pm \sqrt{-(x+y)^2})^6 &= ?\end{aligned}$$

VI. EVOLUTION.

299. To Deduce a Rule for extracting any root of an Imaginary.

From what precedes, (297), we have the following

RULE.—Place the quantity under the form $\sqrt{x}\sqrt{-1}$. Find the required root of $\sqrt{x}$ by the ordinary rules. After this place $\sqrt{-1}$ having for index twice the index of the required root.

$$\begin{aligned}\text{Ex. } \sqrt[10]{\sqrt{-1024}} &= ? & \sqrt[5]{ab\sqrt{-c}} &= ? & \sqrt[7]{abc\sqrt{-(a+x)}} &= ? \\ \sqrt[3]{a^2b^2c\sqrt{-a(x^2+y^2)}} &= ? & \sqrt[m]{a^nx^{2n}\sqrt{-(x+y)^2}} &= ?\end{aligned}$$

NOTE.—The principles now discussed with reference to radicals of the second degree hold, in the case of radicals of any degree, except where the principle involved depends entirely on the fact that the index is 2; *e. g.* (281).

SECTION II.

RADICALS OF HIGHER DEGREES.

(A.) TRANSFORMATIONS.

300. I.—To Deduce a Rule for reducing a Radical of any degree to its simplest form.

Let $a\sqrt[n]{\frac{dc^{n+1}}{x}}$ be any radical of the n th degree that is to be reduced to its simplest form.

Since we desire the quantity under the radical sign in the result to contain no fractions, we transform $\frac{dc^{n+1}}{x}$ so that the denominator may be a perfect n th power, thus, $\frac{dc^{n+1}x^{n-1}}{x^n}$. The radical then becomes $a\sqrt[n]{\frac{dc^{n+1}x^{n-1}}{x^n}}$. Factoring the quantity under the radical sign, we have

$$a\sqrt[n]{dcx^{n-1} \times \frac{c^n}{x^n}}.$$

Here we see that the factor $\frac{c^n}{x^n}$ is the highest perfect n th power in $\frac{dc^{n+1}x^{n-1}}{x^n}$; the other factor, dcx^{n-1} , contains no perfect n th power. By (217)

$$a\sqrt[n]{dcx^{n-1} \frac{c^n}{x^n}} = a\sqrt[n]{dcx^{n-1}} \sqrt[n]{\frac{c^n}{x^n}}; \quad \sqrt[n]{\frac{c^n}{x^n}} = \frac{c}{x} \quad (215).$$

$$\text{Hence } a\sqrt[n]{dcx^{n-1} \frac{c^n}{x^n}} = a \frac{c}{x} \sqrt[n]{dcx^{n-1}} = \frac{ac}{x} \sqrt[n]{dcx^{n-1}}.$$

The original radical then, $a\sqrt[n]{\frac{dc^{n+1}}{x}} = \frac{ac}{x} \sqrt[n]{dcx^{n-1}}$; which result is in its simplest form. As the letters here employed may represent any quantities whatever, we have the following

RULE.—Transform the quantity under the radical sign so that the denominator may be a perfect n th power.

Separate the result into two factors; one the highest perfect n th power in the quantity, the other containing no perfect n th power.

Extract the n th root of the first factor and multiply the result by the given coefficient. Prefix the product to the radical sign (with the original index), under which place the second of the two factors just mentioned.

EXAMPLES.

Reduce the following to their simplest form:

$$\begin{aligned} & 9\sqrt[3]{32a^{10}b^{\frac{5}{3}}c^{\frac{5}{3}}d}; \quad 8\sqrt[6]{.729(x+y)^{\frac{2}{3}}a^6b^{12}cdf^3}; \quad 18\sqrt[3]{a^3b^3(x+y)^{\frac{2}{3}}yz}; \\ & 4\sqrt[4]{ab(x+y)^{2m}c^{\frac{m}{n}}}; \quad x\sqrt[3]{\frac{8a^4}{27b^3} + \frac{16a^2}{27b^3}} = \frac{2ax}{3b}\sqrt[3]{a+2b}; \quad \sqrt[7]{82a^{14}b^8c^{\frac{7}{2}}}; \\ & \sqrt[4]{\frac{(cx-x^2)^{2m}(a+x)^2}{(b+x)^n}} = \frac{(cx-x^2)^{\frac{2}{4}}}{b+x}\sqrt[4]{(a+x)^2}; \quad 3\sqrt[4]{256a^8b^{12}c}; \\ & \sqrt[n]{\frac{(x+y)^{3n}(a+b)^{\frac{n}{2}}c^2d^{-n}}{(x-y)^{n+2}(a+b)^cx}}. \end{aligned}$$

301. II.—To Deduce a Rule for introducing a factor under a radical sign having any index.

Let the given expression be of the form $a\sqrt[n]{b}$, and let it be required to pass a under the radical sign. From the definition of a root it follows that any quantity is the n th power of its own n th root. Hence $a = \sqrt[n]{a^n}$. The expression $a\sqrt[n]{b}$ is therefore equivalent to $\sqrt[n]{a^n}\sqrt[n]{b}$, which (217) is equal to $\sqrt[n]{a^nb}$. Here we see that the factor a is found under the radical sign, with its exponent multiplied by the index of the radical. As the letters employed may represent any quantities whatever, we deduce the following

RULE.—Raise the given factor to the power of which the exponent equals the index of the radical. Multiply the quantity already under the sign by this result.

302. Scholium.—This transformation is employed to obtain the value of a given radical to within 1.

Thus, let it be required to obtain the value of $a\sqrt[n]{b}$. The ordinary rules for evolution give $\sqrt[n]{b}$ to within 1, but on multiplying this result by a , we multiply the error, and our final result may be far from the truth. If, however, we place $a\sqrt[n]{b} = \sqrt[n]{a^n b}$, the ordinary rules for evolution give at once $\sqrt[n]{a^n b}$, and therefore the value of the original expression $a\sqrt[n]{b}$, to within 1.

TO ILLUSTRATE, take $10\sqrt[5]{195}$; $\sqrt[5]{195} = 2$ to within 1. Then if we merely substitute, $10\sqrt[5]{195} = 10 \times 2 = 20$. But $10\sqrt[5]{195} = \sqrt[5]{19500000} = 29$, to within 1. Hence we would have committed an error of about 9 by taking the first result as the value of $10\sqrt[5]{195}$.

303. Scholium II.—Since we may obtain the value (approximate or exact), of any expression of the form $a\sqrt[n]{b}$, if we have exponents of this form we may obtain the value of the expression, and then operate as if the given exponent were rational.

EXAMPLES.

In the following expressions the coefficients are to be introduced under the radical signs.

$$1. 8c\sqrt[3]{126ab^2c}; 3am\sqrt[3]{5ax^2y}; -2a^6x\sqrt[3]{7rsx^2}; a^2b\sqrt[n]{a^{1-2n}b^{1-n}c}; \\ \frac{1}{7}\sqrt[3]{343}; \frac{b}{6}\sqrt[3]{180ax^2}; 3\sqrt[3]{\frac{abc}{3}}; ab^3\sqrt[n]{\frac{a^4}{b^4}}.$$

$$\text{Ans. } \sqrt[3]{8064ab^2c^3}; \sqrt[3]{135a^4m^3x^2y}; \sqrt[3]{-384a^6x^3rs}; \sqrt[n]{abc}; \\ \sqrt[4]{\frac{1}{7}}; \sqrt[3]{\frac{5ab^3x^2}{6}}; \sqrt[3]{9abc}; \sqrt[n]{a^{n+4}b^{3n-4}}.$$

$$2. \frac{1}{a-x}\sqrt[4]{a^4-x^4}; \frac{x}{x^2-y^2}\sqrt[3]{\frac{a(x-y)^4(x+y)^5}{bx(x^2+y^2)}}; 2x^2y\sqrt[n]{256x^3y^2z}; \\ \frac{x^2+2xy+y^2}{x^2-2xy+y^2}\sqrt[6]{\frac{(x-y)^{11}x^2m}{(x+y)^{10}y^2n}}; \frac{-1+\sqrt{5}}{2}\sqrt{\frac{5-\sqrt{5}}{2}}; (1-x)\sqrt[3]{x^2}.$$

Answers to last three:

$$\sqrt[6]{\frac{(x^2+xy)^2m}{(xy^2-y^3)n}}; \sqrt{5-2\sqrt{5}}; \sqrt[3]{x^6-3x^4+3x^2-x^8}.$$

Radicals of the form $\sqrt{5-2\sqrt{5}}$ are called *Double Radicals*.

$$3. \frac{1}{1-x}\sqrt{1-x^4}; \frac{(a-x)^2}{(a+x)^2}\sqrt[n]{\frac{(a^2-x^2)^5}{(a^2+x^2)^5}}; (a-b)\sqrt[n]{(a-b)^{n-2}(a+b)^2}$$

4. Find the approximate values of the following:

$$\frac{1}{2}\sqrt[4]{288}; \frac{1}{3}\sqrt[3]{2700}; \frac{1}{2}\sqrt[5]{13824}; 2\sqrt[7]{\frac{2211}{128}}; 3\sqrt[7]{\frac{2412}{2187}}$$

Ans. 2.06; 4.65; 1.44; 3.00; 3.04.

304. A Rational Quantity may be reduced to the form of a Radical of any required degree by the following

RULE.—Place under a radical sign having the required index that power of the given quantity, of which the exponent is equal to the index of the radical.

Thus, from the definition of a root, $\sqrt[n]{a^n} = a$.

EXAMPLES.

1. Reduce $a-x$ to the form of a radical of the sixth degree; $r+x-s$ to the form of a radical of the third degree.

2. Reduce 6 to the form of a radical of the sixth degree; 2 to the form of a radical of the tenth degree; 3 to the form of a radical of the eighth degree.

305. III.—To Deduce a Rule for reducing a Radical having any index to an equivalent one having a given index.

Let it be required to reduce $\sqrt[6]{a}$ to an equivalent radical having the index 6. Since the number of factors (6) into which the quantity in the new expression is to be divided is twice the number (3) into which a is to be divided, in order that one of the new factors should equal $\sqrt[6]{a}$ it is necessary that the quantity to be factored should have twice as many factors of the form $\sqrt[6]{a}$, as a has. a has three factors of this form; hence the new quantity must have six of them; and since the product of three equals a , the product of six will equal aa , i. e., a^2 . If a^2 , then, be divided into 6 equal factors, any one of these factors will equal $\sqrt[6]{a}$; i. e., $\sqrt[6]{a} = \sqrt[6]{a^2}$.

Again, let it be required to reduce $\sqrt[n]{a}$ to an equivalent radical having the index nx . $\sqrt[n]{a}$ means that a is to be divided into n equal factors. If the index of the radical be nx , the quantity under the radical sign is to be divided

into nx factors, of which any one is to be equal to one of the n factors of a . Since each of the nx factors equals $\sqrt[n]{a}$, the product of n of them equals a . Hence, if we divide the nx factors into x groups, of n factors each, the product of all the factors in each group will be a . There being x groups, each equal to a , the whole product will be equal to a taken x times as a factor, i. e., to a^x . Hence, $\sqrt[nx]{a^x} = \sqrt[n]{a}$, is the required radical. Since the letters here employed represent any quantities whatever, we deduce the following

RULE.—*Divide the new index by the old. Raise the quantity given under the radical sign to the power indicated by this quotient, and place the result under the radical sign with the new index.*

306. Scholium I.—*The use of fractional exponents enables us to arrive at the same result very expeditiously.*

$$\text{Thus, } \sqrt[n]{a} = a^{\frac{1}{n}} = a^{\frac{x}{nx}} \quad (182). \quad a^{\frac{x}{nx}} = \sqrt[nx]{a^x}.$$

$$\text{Again, } \sqrt[n]{a} = a^{\frac{1}{n}} = a^{\frac{m}{nm}} = a^{\frac{m}{n} \cdot \frac{1}{m}} = \left(a^{\frac{m}{n}}\right)^{\frac{1}{m}} \quad (195)$$

$$\left(a^{\frac{m}{n}}\right)^{\frac{1}{m}} = \sqrt[m]{a^{\frac{m}{n}}} \text{ i. e., } \sqrt[n]{a} = \sqrt[m]{a^{\frac{m}{n}}}.$$

307. Corollary.—*Since $\sqrt[n]{a} = \sqrt[nx]{a^x}$, we see that we may multiply or divide the index of a radical by any quantity, provided we multiply or divide the exponent of the quantity under the radical sign by the same multiplier or divisor.*

308. Scholium II.—By (307) we are often enabled to reduce the degree of radicals of special forms. Whenever the quantity under the radical sign is a perfect power of which the exponent is an exact divisor of the index of the root, we may divide the index of the root by this exponent, and extract the corresponding root of the quantity under the radical sign.

$$\text{Thus, } \sqrt[6]{a^2} = \sqrt[3]{a}, \quad \sqrt[12]{x^4} = \sqrt[3]{x}.$$

EXAMPLES.

1. Transform $\sqrt[3]{\frac{3}{5}}$ into a radical of the 6th degree.

Ans. $\sqrt[6]{\frac{27}{125}}$.

2. Transform $\sqrt[3]{\frac{5}{7}}$, $\sqrt[6]{\frac{2}{3}}$, $\sqrt[4]{\frac{a}{b}}$ and $\sqrt{\frac{a-x}{a+x}}$ into radicals of the 12th degree.

3. Transform $\sqrt[3]{\frac{a}{b}}$, $\sqrt[5+1]{\frac{a-x}{a^2+x^2}}$, $2\sqrt[5]{\frac{7(x^2-y^2)}{8(x^2+y^2)}}$ and $\sqrt[5+5]{\frac{1}{a+x}}$ into radicals of the r th degree, r being equal to $10n+10$.

4. Transform $\sqrt[r]{-\frac{ax}{b}}$, $\sqrt[3r]{\frac{a-x}{b+x}}$, $(a+b)\sqrt[5r]{\frac{a^2+bx}{a^2-bx}}$ and $3\sqrt[n]{m}$ into radicals of the 7rth degree.

309. IV.—To Deduce a Rule for reducing Radicals having different indices to equivalent Radicals having a common index.

Having several radicals with different indices we may assume any quantity for a common index, and reduce each radical to an equivalent radical having this index by (305). To avoid fractional exponents under the new radical signs we assume, for the common index, a quantity exactly divisible by each of the given indices. Such a quantity is obviously a common multiple of the given indices. We assume the lowest common multiple in order that the common index may be as low as possible.

To ILLUSTRATE: Take $\sqrt[6]{a}$, $\sqrt[4]{b}$, $\sqrt[3]{c}$. To avoid fractional exponents under the new radical signs we assume for a common index a common multiple of 6, 4, and 3. That the resulting radicals should be as low in degree as possible, we assume the lowest common multiple of 6, 4, and 3, viz., 12. Reducing (305) each radical to an equivalent radical with the index 12, we have $\sqrt[12]{a^2}$, $\sqrt[12]{b^3}$, $\sqrt[12]{c^4}$.

From the foregoing considerations we deduce the following

RULE.—Reduce each of the given radicals to an equivalent radical having as index the lowest common multiple of the given indices.

EXAMPLES.

Reduce, in each example, the radicals to equivalent radicals having a common index:

1. $\sqrt[3]{5}$, $4\sqrt[3]{7}$, $(a+b)\sqrt[4]{\frac{(a+b)^3}{(a-b)^3}}$, $\frac{1}{x-y}\sqrt{x-y}$, and $\frac{1}{7}\sqrt[7]{343}$.
2. $(a+b)\sqrt[2]{\frac{(a-b)^{n+1}}{(a^2-b^2)^{n+2}}}$, $5\sqrt[2n]{\frac{r+s}{s-x}}$, $\sqrt[2n+1]{\frac{1}{1-x+x^2}}$ and $a\sqrt[2n+3]{\frac{1-x}{1+x}}$.
3. $\sqrt[2n+2]{\frac{x-a}{x+a}}$, $\sqrt[2n+1]{\frac{x+a}{x-a}}$, $\frac{1}{x-a}\sqrt[2(n+2)]{\frac{x^2-a^2}{x^2+a^2}}$ and $\sqrt[2n+2n+2n]{\frac{a+bx+ax}{d+rx+sz}}$.
4. $\sqrt[5]{\frac{a}{a+x}}$, $\sqrt[3]{\frac{1}{1-x}}$, $\sqrt{\frac{1+x}{1+x^2}}$, $\sqrt[10]{a-x}$ and $\sqrt[6]{a^2b-b^2a}$.

(B.) OPERATIONS ON RADICALS OF ANY DEGREE.

Having shown how to reduce radicals with different indices to equivalent radicals having the same index, we shall assume in every discussion that all radicals given have been subjected to this transformation.

I. ADDITION.

310. To Deduce a Rule for adding Radicals of any degree.

I. Similar Radicals.

Let it be required to add $a_1\sqrt[n]{x}$, $b_1\sqrt[n]{x}$, and $c_1\sqrt[n]{x}$, a , b , and c being any coefficients whatever, positive or negative, numerical or literal, entire or fractional.

Since $\sqrt[n]{x}$ is taken a times in the first term, b times in the second, and c times in the third (additively or subtractively, as the case may be), it will be taken $a+b+c$ times if all the terms are taken together, i. e., if the aggregate of the

terms be taken. Hence $a\sqrt[n]{x} + b\sqrt[n]{x} + c\sqrt[n]{x} = (a+b+c)\sqrt[n]{x}$. Since the letters employed may represent any quantities whatever, we deduce the following

RULE.—Take the algebraic sum of the coefficients. Prefix this sum, as a coefficient, to the common radical part.

We can treat this case by the ordinary rules for addition, by placing the radical part in each term equal to any quantity, as z , and then substituting it in the sum for the value of z .

Thus $5\sqrt[n]{a} + 6\sqrt[n]{a} + 2\sqrt[n]{a} = ?$ Denote $\sqrt[n]{a}$ by z . Then we have, $5z + 6z + 2z = 13z$; and substituting for z its value, $\sqrt[n]{a}$, we obtain $13\sqrt[n]{a}$ as the sum required.

II. Dissimilar Radicals.

Since we cannot add things of different kinds we cannot add dissimilar radicals. If we have several given to be added, then, we simply connect them by their proper signs.

EXAMPLES.

$$1. \quad 3\sqrt[4]{a^2b} + 5\sqrt[4]{a^2b} + 6\sqrt[4]{a^2b} + 3ac\sqrt[4]{a^2b} = ?$$

$$\text{Ans. } (9+8ac)\sqrt[4]{a^2b}.$$

$$2. \quad \text{Add } 5\sqrt[3]{729a^3b^3c^3} \text{ and } 7ab\sqrt[3]{b^3c^3}. \quad \text{Ans. } 52ab\sqrt[3]{b^3c^3}.$$

$$3. \quad \text{Add } 2a\sqrt[4]{48a^3b^3c^3}, a^2c\sqrt[4]{243a^2b^3c^3} \text{ and } \sqrt[4]{1875a^{10}b^3c^3}.$$

$$\text{Ans. } 12a^2c\sqrt[4]{3a^4b^3c^3}.$$

$$4. \quad \text{Add } \sqrt[3]{24a^3b^3c}, 4b^2\sqrt[3]{3a^3c} \text{ and } 5a\sqrt[3]{51cb^3}.$$

$$5. \quad \text{Add } x\sqrt[n]{a^{2m}(x+y)^mc} \text{ and } xa\sqrt[n]{a^m(x-y)^mc}. \quad \text{Ans. } 2a^2x^2\sqrt[n]{c}$$

$$6. \quad \text{Add } x\sqrt{1+\left(\frac{y}{x}\right)^{\frac{1}{2}}} \text{ and } y\sqrt{1+\left(\frac{x}{y}\right)^{\frac{1}{2}}}.$$

$$\text{Ans. } (x^{\frac{3}{4}}+y^{\frac{3}{4}})\sqrt{x^{\frac{1}{2}}+y^{\frac{1}{2}}}.$$

$$7. \quad \text{Add } x\sqrt[n]{1+\left(\frac{y}{x}\right)^{\frac{1}{m}}} \text{ and } y\sqrt[n]{1+\left(\frac{x}{y}\right)^{\frac{1}{m}}}.$$

Introducing the coefficients, we obtain

$$\begin{aligned} & \sqrt[n]{x^m + \frac{x^m y^{\frac{1}{m}}}{x^{\frac{1}{m}}}} + \sqrt[n]{y^m + \frac{y^m x^{\frac{1}{m}}}{y^{\frac{1}{m}}}} = \sqrt[n]{x^m + x^{m-\frac{1}{n}} y^{\frac{1}{n}}} + \sqrt[n]{y^m + y^{m-\frac{1}{n}} x^{\frac{1}{n}}} \\ &= \sqrt[n]{x^{m-\frac{1}{n}} (x^{\frac{1}{n}} + y^{\frac{1}{n}})} + \sqrt[n]{y^{m-\frac{1}{n}} (y^{\frac{1}{n}} + x^{\frac{1}{n}})} = \\ & \quad (x^{1-\frac{1}{m}} + y^{1-\frac{1}{m}})^{\frac{1}{n}} \sqrt[n]{x^{\frac{1}{n}} + y^{\frac{1}{n}}}. \end{aligned}$$

8. Add $y\sqrt[n]{1 + \left(\frac{x}{y}\right)^{\frac{m}{n}}}$ and $x\sqrt[n]{1 + \left(\frac{y}{x}\right)^{\frac{m}{n}}}$. We have,

$$y\sqrt[n]{1 + \left(\frac{x}{y}\right)^{\frac{m}{n}}} = y\sqrt[n]{\frac{y^{\frac{m}{n}} + x^{\frac{m}{n}}}{y^{\frac{m}{n}}}} = y^{1-\frac{1}{n}} \sqrt[n]{y^{\frac{m}{n}} + x^{\frac{m}{n}}}. \quad \text{Similarly}$$

$$x\sqrt[n]{1 + \left(\frac{y}{x}\right)^{\frac{m}{n}}} = x^{1-\frac{1}{n}} \sqrt[n]{y^{\frac{m}{n}} + x^{\frac{m}{n}}}. \quad \text{Hence the sum is}$$

$$\left(x^{\frac{n-1}{n}} + y^{\frac{n-1}{n}}\right)^{\frac{1}{n}} \sqrt[n]{x^{\frac{m}{n}} + y^{\frac{m}{n}}}.$$

9. Add $(x+y)\sqrt[n]{a^2(x+y)^{3n}}$ and $(x-y)\sqrt[n]{a^2(x-y)^{3n}}$.

$$\text{Ans. } 2x(x^2+3y^2)\sqrt[n]{a^2}.$$

10.

$$\text{Add } \sqrt[n]{c^m a^{mp+3b^{mn+5}}}, \sqrt[n]{r^m a^{3m-mn+3b^{m+5}}} \text{ and } -\sqrt[n]{s^{3m} a^{m+3b^5}}.$$

$$\text{Ans. } (capb^n + ra^{3-m}b - s^3a)\sqrt[n]{a^3b^5}.$$

II. SUBTRACTION.

311. To Deduce a Rule for subtracting one Radical, of any degree whatever, from another.

I. Similar Radicals.

Let it be required to subtract $a\sqrt[n]{b}$ from $c\sqrt[n]{b}$. Since the remainder, added to the subtrahend, must produce the minuend, the radical part of the remainder must be the common radical part of the given quantities, and the coefficient of the remainder must be such that, added to the coefficient of the subtrahend, it will produce the coefficient of

the minuend; i. e., it must be the algebraic difference (found by the ordinary rules), between the coefficient of the minuend and the coefficient of the remainder. In the given case, then, the remainder is $(c-a)\sqrt[n]{b}$.

From the foregoing considerations we deduce the following

RULE.—*Change the sign of the subtrahend, or conceive it to be changed, and proceed as in addition.*

The same result may be obtained much more expeditiously by the use of fractional exponents.

II. Dissimilar Radicals.

For reasons previously given, the subtraction can (in this case) be only indicated. Thus, if the difference between $a\sqrt[n]{b}$ and $c\sqrt[n]{d}$ is required, a quantity is to be found, which, added to $c\sqrt[n]{d}$ will give $a\sqrt[n]{b}$ as a sum. $a\sqrt[n]{b}-c\sqrt[n]{d}$ is evidently such a quantity. This expression is, however, merely the indication of the operation to be performed, and might have been obtained directly by the foregoing rule.

EXAMPLES.

1. From $\sqrt[5]{128}$ subtract $-8\sqrt[5]{1458}$. *Rem.* $26\sqrt[5]{2}$.

2. From $3\sqrt[3]{8a^3b+16a^4}$ subtract $3\sqrt[3]{b^4+2ab^3}$.
Rem. $3(2a-b)\sqrt[3]{2a+b}$.

3. From $\left(\frac{f}{d} + \frac{g^2}{3cd}\right)\sqrt[6]{\frac{24c^3d^2}{f^2g}}$ subtract $\sqrt[6]{\frac{8g^{11}}{243c^3d^4f^2}}$.
Rem. $\sqrt[6]{\frac{24c^3f^4}{d^4g}}$

4. From $\sqrt[5]{\frac{1}{a^2}} + \sqrt[4]{\frac{1}{a^3}}$ subtract $-\left(\sqrt[3]{\frac{1}{a^2}} + \sqrt{\frac{1}{a}}\right)$.
Rem. $\frac{1 + \sqrt[20]{a} + \sqrt[15]{a^2} + \sqrt[10]{a^3}}{\sqrt[5]{a^4}}$.

5. From $\frac{1}{s}\sqrt{\frac{1}{y}} + \frac{x}{y}\sqrt[3]{\frac{1}{z}}$ subtract $-\frac{y}{x}\sqrt{\frac{1}{z}}$.
Rem. $\frac{x\sqrt{y} + x^3\sqrt[3]{z} + y^2\sqrt{z}}{xyz}$.

$$6 \text{ From } \sqrt[4]{\frac{324}{2304}} \text{ subtract } \sqrt[4]{\frac{8}{421875}} \quad \text{Rem. } \frac{11}{60}\sqrt[4]{6}.$$

$$7. \text{ From } \frac{2a^2}{x^2} \text{ subtract } \frac{a}{a-\sqrt{a^2-x^2}} \quad \text{Rem. } \frac{a}{a+\sqrt{a^2-x^2}}$$

$$8. \text{ From } \frac{\sqrt{x^2+1}+\sqrt{x^2-1}}{\sqrt{x^2+1}-\sqrt{x^2-1}} \text{ subtract } \frac{\sqrt{x^2-1}-\sqrt{x^2+1}}{\sqrt{x^2+1}+\sqrt{x^2-1}}.$$

III. MULTIPLICATION.

312. To Deduce a Rule for multiplying Radicals of any degree.

Let it be required to multiply any radical of the n th degree, as $a\sqrt[n]{b}$, by any other, as $c\sqrt[n]{d}$. Now $a\sqrt[n]{b} = \sqrt[n]{a^n b}$ and $c\sqrt[n]{d} = \sqrt[n]{c^n d}$. Hence the multiplication may be indicated thus: $\sqrt[n]{a^n b} \cdot \sqrt[n]{c^n d}$, or (217) $\sqrt[n]{a^n c^n b d}$; simplifying which, we have $ac\sqrt[n]{bd}$. The coefficient of this product is the product of the coefficients of the factors. The quantity under the radical sign in the product is the product of the quantities under the signs in the given factors. The index of the root is unchanged.

As the letters employed may represent any quantities whatever, we deduce the following

RULE.—*Multiply together the coefficients of the factors. Prefix the result, as a coefficient, to the radical sign (with the index unchanged) under which place the product of the quantities under the signs in the two factors.*

313. Scholium.—This rule is evidently applicable to the operation of finding the *continued product of any number of factors.*

EXAMPLES.

$$1. \text{ Multiply together } 3\sqrt[5]{2xy}, 4\sqrt[3]{5xy}, \text{ and } 12ac\sqrt[4]{3bx}.$$

$$2. \text{ Multiply } 3\sqrt[3]{\frac{2}{3}} \text{ by } 6\sqrt[5]{\frac{1}{2}} \quad \text{Ans. } 6\sqrt[15]{236196}.$$

$$3. \text{ Multiply } \sqrt[3]{8-3\sqrt{5}} \text{ by } \sqrt[3]{8+3\sqrt{5}} \quad \text{Ans. } \sqrt[3]{19}.$$

4. Multiply

$$\sqrt[3]{a+b} + \sqrt[3]{a-b} \text{ by } \sqrt[3]{a^3-b^3} - \sqrt[3]{(a+b)^3} - \sqrt[3]{(a-b)^3}.$$

Ans. $-2a$.

$$5. \sqrt[4]{\frac{2}{3}} \cdot \sqrt[6]{\frac{3}{2}} \cdot \sqrt[5]{6} = ? \quad \sqrt[4]{5} \cdot \sqrt[6]{1} = ? \quad \sqrt[5]{1-x^3} \cdot \sqrt[3]{\frac{1}{1-x}} = ?$$

$$\sqrt[3]{xy} \cdot \sqrt[12]{x^{-1}y^{-6}} = ? \quad \text{Ans. } \sqrt[12]{96}; \sqrt[12]{5}; \sqrt[6]{\frac{1+x}{1-x}}; \sqrt[12]{\frac{x^3}{y}}.$$

Reduce each result to its simplest form.

$$6. \left(\frac{2}{3} \sqrt[3]{\frac{1}{b}} \sqrt[3]{a} - \sqrt[3]{a} - \frac{1}{6} \sqrt[3]{ab^3} \right) \cdot \left(\sqrt[3]{\frac{a^3}{b^3}} + \frac{3}{2} \sqrt[3]{\frac{1}{b}} \sqrt[3]{a} - 6 \sqrt[3]{a} \right) = ?$$

$$\text{Ans. } \frac{2\sqrt[3]{a^7}}{3b} - \frac{17}{3} \sqrt[3]{\frac{1}{b}} \sqrt[3]{a^3} + \frac{23}{4} \sqrt[3]{a^3} + \sqrt[3]{a^6 b^3}.$$

$$7. \text{ Multiply } \frac{\sqrt[5]{a^5}}{6} + \frac{2\sqrt[3]{a^2}}{5} + \frac{4\sqrt{a}}{5}$$

$$\text{by } \frac{5\sqrt[5]{a^3}}{6} - \frac{2\sqrt[3]{a^2}}{5} + \frac{4\sqrt{a}}{5}$$

$$\frac{5\sqrt[5]{a^5}}{36} + \frac{1}{3} \sqrt[5]{a^2} + \frac{2}{3} \sqrt[5]{a^3}$$

$$- \frac{1}{3} \sqrt[5]{a^3} - \frac{4}{5} \sqrt[3]{a^4} - \frac{8}{5} \sqrt[5]{a^7}$$

$$+ \frac{4}{30} \sqrt[5]{a^5} + \frac{8}{25} \sqrt[5]{a^7} + \frac{16a}{25}$$

$$\frac{5}{36} \sqrt[5]{a^3}$$

$$- \frac{32}{25} \sqrt[5]{a^7} + \frac{16}{25} a \quad \text{Ans.}$$

$$8. \text{ Multiply } 4\sqrt{x} - 2\sqrt[3]{xy} + 2\sqrt[4]{x^2y^3} - 3\sqrt[5]{xy^3}$$

$$\text{by } \frac{1}{2} \sqrt[3]{x^3} - \frac{1}{3} \sqrt[5]{x^3y} + \frac{1}{4} \sqrt[6]{x^2} - \frac{1}{4} \sqrt{x^3}$$

$$9. \left(4\sqrt[3]{\frac{1}{a}} \sqrt[3]{b^4} - \frac{b}{2} + \frac{2}{9} \sqrt[3]{a^3b^4} \right) \cdot \left(\frac{5a\sqrt[3]{b}}{6} - \frac{2}{3} \sqrt[3]{a^3} - \frac{3a^3}{4} \cdot \sqrt[3]{\frac{1}{b}} \right) = ?$$

$$\text{Ans. } \frac{10}{3} \cdot \sqrt[5]{a^3b^{10}} - \frac{37}{12} a \sqrt[3]{b^4} - \frac{67}{27} b \sqrt[3]{a^3} + \frac{49}{216} a^2 \sqrt[3]{b^3} - \frac{1}{6} \sqrt[5]{a^{10}b^3}.$$

IV. DIVISION.

314. To Deduce a Rule for the division of Radicals of any degree.

Let it be required to divide $a\sqrt[n]{b}$ by $x\sqrt[n]{y}$. $a\sqrt[n]{b} = \sqrt[n]{a^n b}$; $x\sqrt[n]{y} = \sqrt[n]{x^n y}$. Hence the division of $a\sqrt[n]{b}$ by $x\sqrt[n]{y}$ may be indicated thus: $\frac{\sqrt[n]{a^n b}}{\sqrt[n]{x^n y}} = \sqrt[n]{\frac{a^n b}{x^n y}}$ (217).

Simplifying this radical, we have $\frac{a}{x}\sqrt[n]{\frac{b}{y}}$. The coefficient of the quotient is the quotient obtained by dividing the coefficient of the dividend by the coefficient of the divisor, and the quantity under the radical sign is the quotient of the quantity under the sign in the dividend by the corresponding quantity in the divisor. As the letters employed may represent any quantities whatever, we deduce the following

RULE.—*Divide the coefficient of the dividend by the coefficient of the divisor. To the result, as a coefficient, annex the radical sign, with unchanged index, placing under it the quotient obtained by dividing the quantity under the radical sign in the dividend by the corresponding quantity in the divisor.*

EXAMPLES.

$$1. \frac{36\sqrt[3]{a^3 b}}{12\sqrt[3]{ab^3}} = ? \quad \sqrt[4]{\frac{a}{b}} \div \sqrt[4]{\frac{a}{b}} = ? \quad \frac{\sqrt[3]{8} + \sqrt[3]{12} + \sqrt[3]{2}}{2\sqrt[3]{2}} = ?$$

$$\frac{4\sqrt[5]{12}}{2\sqrt[5]{3}} = ? \quad \sqrt[5]{a^5 b^{n-1} c^3 d} \div \sqrt[5]{\frac{d c^{n-1}}{a^3 b^4}} = ?$$

$$\text{Ans. } 3\sqrt[12]{a^3 b^3}; \sqrt[4]{\frac{b}{a}}; 1 + \frac{\sqrt[3]{18}}{2} + \frac{\sqrt[3]{8}}{4}; 2\sqrt[3]{\frac{16}{3}}; \sqrt[5]{\frac{a^{n+3} b^{n+3}}{c^{n-2}}}$$

$$2. \frac{8\sqrt[5]{\frac{2}{3}}}{4\sqrt[4]{\frac{2}{3}}} = ? \quad \frac{15\sqrt[3]{a^3 - 2ab + b^3}}{3\sqrt[3]{a^3 - b^3}} = ? \quad \frac{2132\sqrt[5]{a^5 b^5 c^{n-1}}}{4\sqrt[5]{a^3 b^3 c^2}} = ?$$

$$3. \frac{\sqrt[n]{b^3}}{\sqrt[n]{c}\sqrt[n]{d}\sqrt[n]{a^3}} \div \frac{\sqrt[n]{d^3a^{20}}}{\sqrt[n]{b^3}\sqrt[n]{a^{20}}} = ? \quad \frac{1}{a\sqrt[n]{a^3}} \div \frac{1}{\sqrt[n]{a^{18}}} = ?$$

$$a^3\sqrt[n]{\frac{b^4}{ac^3}} \div \sqrt[n]{a^{15}}\sqrt[n]{b} = ?$$

$$4. \text{ Divide } \sqrt[n]{x^3} - \sqrt[n]{y^3} \text{ by } \sqrt[n]{x} - \sqrt[n]{y}.$$

$$\begin{array}{r} \sqrt[n]{x^3} - \sqrt[n]{y^3} \quad \bigg| \quad \sqrt[n]{x} - \sqrt[n]{y} \\ \hline \sqrt[n]{x^3} - \sqrt[n]{x^2y} \quad \bigg| \quad \sqrt[n]{x^3} + \sqrt[n]{xy} + \sqrt[n]{y^3} \quad \text{Quotient.} \\ \hline \sqrt[n]{x^2y} - \sqrt[n]{y^3} \\ \sqrt[n]{x^2y} - \sqrt[n]{xy^2} \\ \hline \sqrt[n]{xy^2} - \sqrt[n]{y^3} \\ \sqrt[n]{xy^2} - \sqrt[n]{y^3} \\ \hline 0 \end{array}$$

$$5. \frac{\sqrt[n]{a^3} - a^3\sqrt[n]{\frac{1}{b^3}} - b^3\sqrt[n]{a} + \sqrt[n]{b}}{\sqrt[n]{a} - \sqrt[n]{\frac{1}{b^3}}} = ? \quad \frac{\sqrt[n]{a^3} - \sqrt[n]{b^3}}{\sqrt[n]{a} - \sqrt[n]{b}} = ?$$

$$\frac{2\sqrt{ab} + b + a - c}{\sqrt{a} + \sqrt{b} + \sqrt{c}} = ? \quad \text{Ans. to last } \sqrt{a} + \sqrt{b} - \sqrt{c}.$$

$$6. \left(\frac{4b^3}{a^2}\sqrt[3]{\frac{1}{c^3}} - \frac{5}{b}\sqrt[3]{c^3} + \frac{a^3c^3}{b^3} \right) \div \left(2b\sqrt[3]{\frac{1}{a^3}}\sqrt[3]{\frac{1}{c^3}} + 3\sqrt[3]{\frac{1}{a}} + \frac{\sqrt[3]{a^3c^3}}{b} \right) = ?$$

$$\text{Ans. } \frac{2\sqrt[3]{a}}{a} - \frac{3\sqrt[3]{a^3c^3}}{b} + \frac{\sqrt[3]{a^3c^3}}{b^3}.$$

$$7. \frac{a-b}{\sqrt[n]{a} - \sqrt[n]{b}} = ? \quad \frac{a+b}{\sqrt[n]{a} + \sqrt[n]{b}} = ? \quad \frac{a-b}{\sqrt[n]{a} + \sqrt[n]{b}} = ?$$

$$\text{Ans. to first } \sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} + \dots + \sqrt[n]{b^{n-1}}.$$

V. INVOLUTION.

315. To Deduce a Rule for raising a Radical to any power.

Let $a\sqrt[n]{b}$ represent any radical, and m the exponent of the required power. To raise $a\sqrt[n]{b}$ to the m th power is to take it m times as a factor, that is,

$$(a\sqrt[n]{b})^m = a\sqrt[n]{b} \cdot a\sqrt[n]{b} \cdot a\sqrt[n]{b} \dots$$

to m factors. This product is obtained by taking the continued product of the coefficients, and multiplying it into the n th root of the continued product of the quantities under the radical sign. The product obtained by taking a m times as a factor, is a^m . The product obtained by taking b m times as a factor is b^m . Hence $(a\sqrt[n]{b})^m = a^m\sqrt[n]{b^m}$. From the general nature of the reasoning we have the following

RULE.—*Raise the coefficient of the given radical to the required power. Annex to the result the radical sign with index unchanged, placing under it the required power of the quantity under the sign in the given expression.*

316. Scholium.—If n is a multiple of m , we may simplify $\sqrt[n]{b^m}$ by (308). Hence, to raise a radical to a power of which the exponent is an exact divisor of the index of the given radical, we have the

RULE.—*Divide the index of the radical by the exponent of the power, leaving the quantity under the sign unchanged. To this prefix as a coefficient the required power of the given coefficient.*

VI. EVOLUTION.

317. To Deduce a Rule for extracting any root of a Radical of any degree.

Let it be required to extract the m th root of $a^m\sqrt[n]{b^m}$. Since the m th root of the given radical must, when raised to the m th power, produce the radical itself, the m th power of the coefficient of the root must be the coefficient of the given quantity, and the m th power of the quantity under the radical sign in the root must be the quantity under the radical sign in the given expression, the index in the two cases being the same (315). Hence, to extract any root of a radical of any degree, we have the following

RULE.—*Extract the required root of the coefficient and the same root of the quantity under the radical sign, for the corresponding parts of the root, leaving the index of the radical unchanged.*

318. Scholium I.—If the required root of the quantity under the radical sign cannot be extracted, we use the principle (230) that $\sqrt[n]{\sqrt[n]{x}} = \sqrt[n^m]{x}$, and have the following

RULE.—Extract the required root of the coefficient. After the result place the radical sign, with an index equal to the product of its original index and the index of the required root, leaving the quantity under the radical sign unchanged.

319. Scholium II.—If the required root of neither the coefficient nor the quantity under the radical sign can be extracted, pass the coefficient under the sign, and proceed by (317), considering the transformed radical as having the coefficient 1.

EXAMPLES.

$$1. (5ab\sqrt[4]{b^3c})^3 = ? \quad (a^2\sqrt[3]{rst})^2 = ? \quad (a^2b\sqrt[3]{a^2bc(x+y)^{\frac{1}{mz}-\frac{1}{n}}})^r = ?$$

$$\text{Ans. } 125a^3b^4\sqrt[3]{b^3c^3}; \quad a^2\sqrt[6]{r^2s^2t^2}; \quad a^{m+2r}b^r\sqrt[3]{b^rc^r(x+y)^{\frac{r}{mz}-\frac{r}{n}}}.$$

$$2. (6ab\sqrt[5]{a^3b^3})^2 = ? \quad (am\sqrt[5]{2ab})^{10} = ? \quad (2rs\sqrt[3]{a+b})^{2n} = ?$$

$$\left(\frac{5rst}{a+b}\sqrt[3]{a^2-b^2}\right)^n = ? \quad (\sqrt[3]{x} + \sqrt[3]{y})^2 = ? \quad (\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z})^2 = ?$$

$$3. \left(\sqrt{\frac{a^4-x^4}{a^2}}\sqrt[3]{\frac{a^2m}{a^3-x^3}}\right)^3 = ? \quad ((a-x)\sqrt{a-x})^2 = ? \quad (\sqrt[4]{a+\sqrt{b}})^{12} = ?$$

$$\text{Ans. } \frac{m}{a(a^4+x^4)}\sqrt[3]{a^4-x^4}; \quad (a-x)^{12}; \quad a^3+3a^2\sqrt{b}+3ab+\sqrt{b^3}.$$

$$4. (\sqrt[n]{(a\pm\sqrt{b})c})^{5n} = ? \quad \left(\sqrt[3]{\frac{a^3-2ax+x^3}{a^2+2ax+x^2}}\right)^{12} = ? \quad ((a+x)\sqrt[4]{\frac{a-x}{a+x}})^8 = ?$$

$$(\sqrt[3]{x} + \sqrt[3]{y} - \sqrt[3]{z})^3 = ? \quad (\sqrt{a+x} - \sqrt{a-x})^2 = ?$$

$$5. \sqrt[9]{81a^4} = ? \quad \sqrt[3]{\sqrt[10]{27a^3}} = ? \quad \sqrt[5]{\sqrt[12]{32(x+y)^5c^{10}}} = ?$$

$$\sqrt[n]{\sqrt[m]{a^mb^{2m}(x+y)^{-\frac{m}{r}}}} = ?$$

$$\text{Ans. } \sqrt[9]{9a^2}; \quad \sqrt[10]{3a}; \quad \sqrt[12]{2(x+y)c^2}; \quad \sqrt[n]{ab^2(x+y)^{-\frac{1}{r}}}.$$

$$6. \sqrt[n]{\frac{a}{b}} \sqrt[n]{\frac{b}{a}} = ? \quad \sqrt[n]{(a-b)} \sqrt[n]{a-b} = ? \quad \sqrt[n+1]{(a+b+c)} \sqrt[n]{a+b+c} = ?$$

$$\sqrt[n]{\frac{1}{7}} \sqrt[n]{843} = ? \quad \left(\frac{m \pm \sqrt{x^2 - m^2}}{a} \right)^{\frac{1}{n}} = ?$$

$$7. \sqrt[4]{a^2 - 4\sqrt{a^2} \sqrt[3]{b} + 6a\sqrt[3]{b^2} - 4\sqrt{ab} + \sqrt[3]{b^4}} = ?$$

$$\text{Ans. } \sqrt{a} - \sqrt[3]{b}.$$

$$8. \sqrt[3]{1 - 6\sqrt[4]{x} + 12\sqrt[4]{x} - 8\sqrt[4]{x^3}} = ?$$

$$\sqrt[6]{x^2 - 6x\sqrt[5]{x^2} + 15x\sqrt[5]{x} - 20x + 15\sqrt[5]{x^3} - 6\sqrt[5]{x} + 1} = ?$$

$$9. \sqrt[3]{(8x\sqrt[5]{x^3} + 56x + x\sqrt[5]{x^3} + 28x\sqrt[5]{x} + 28\sqrt[5]{x^3} + 70\sqrt[5]{x^4} + 56x\sqrt[5]{x^2} + 1 + 8\sqrt[5]{x})} = ?$$

$$\text{Ans. } \pm(\sqrt[5]{x} + 1).$$

IMAGINARY QUANTITIES OF ANY DEGREE.

MONOMIALS.

320. Lemma.—*Every imaginary radical may be reduced to the form $c\sqrt[n]{-1}$, c being rational or irrational, but real.*

Let $\sqrt[n]{-p}$ be any imaginary radical. Place $\sqrt[n]{-p} = r$ (1); whence $\sqrt[n]{-1} \times p = r$ (2). Raising both equals to the $2n$ th power, $-1 \times p = r^{2n}$ (3). Since we extract the $2n$ th root of a product by dividing the exponent of each factor by $2n$, i. e., extracting the $2n$ th root of each factor, extracting the $2n$ th root of the equals, (2), we have

$$\sqrt[n]{-1} \cdot \sqrt[n]{p} = r. \therefore (1), \sqrt[n]{-p} = \sqrt[n]{-1} \sqrt[n]{p}.$$

Placing $\sqrt[n]{p} = c$, we obtain $\sqrt[n]{-p} = c\sqrt[n]{-1}$, the form required.

321. Corollary I.—*The product of any root of a positive quantity and the same root of a negative quantity equals the indicated root of the product of the quantities.*

$$\text{Thus, } \sqrt[n]{a} \cdot \sqrt[n]{-y} = \sqrt[n]{a} \sqrt[n]{-1} \sqrt[n]{y} = \sqrt[n]{-1} \sqrt[n]{ay} = \sqrt[n]{-ay}.$$

322. Corollary II.—*Any root of a quantity, divided by the same root of a quantity with an opposite sign, equals the indicated root of the quotient obtained by dividing the first quantity by the second.*

$$\begin{aligned} \text{Thus, } \sqrt[n]{a} \div \sqrt[n]{y} &= \sqrt[n]{a} + \sqrt[n]{y} \sqrt[n]{-1} = \\ \frac{\sqrt[n]{a}}{\sqrt[n]{y}} \times \frac{1}{\sqrt[n]{y}} &= \frac{\sqrt[n]{a}}{\sqrt[n]{y}} \frac{\sqrt[n]{1}}{\sqrt[n]{y}} = \sqrt[n]{\frac{a}{y}} \sqrt[n]{\frac{1}{-1}} = \sqrt[n]{-\frac{a}{y}}. \end{aligned}$$

323. Corollary III — *Imaginary radicals may be transformed by the rules already deduced.*

OPERATIONS ON IMAGINARY QUANTITIES OF ANY DEGREE.

I. ADDITION.

324. To Deduce Rules for adding Imaginary quantities of any degree.

I. Similar Radicals.

From the nature of coefficients and the reasoning in (310) we see that if several terms have a common radical part, even if this part be imaginary, the algebraic sum of the coefficients will show how often this radical part is to be taken, additively or subtractively, in the aggregate of these terms. Hence the rule (310) already deduced for the addition of radicals applies, without change, to the case of imaginary quantities.

II. Dissimilar Radicals.—In the case of these radicals the addition can only be indicated.

II. SUBTRACTION.

325. To Deduce a Rule for the subtraction of Imaginary quantities.

From the definition of coefficients, and (311), it will be seen that the rule of (311) may be applied, without change, to the subtraction of imaginary quantities.

EXAMPLES.

1. $ax^4\sqrt[4]{-b} + (a+c)\sqrt[4]{-bc^4x^8} + (a-c)\sqrt[4]{-bc^4x^8} = ?$
Ans. $ax^4(1+2c)\sqrt[4]{-b}.$
2. $5ab\sqrt[6]{-64abc^6} + 3rs\sqrt[6]{-ab} - 4abc\sqrt[6]{-ab} = ?$
Ans. $3(2abc+rs)\sqrt[6]{-ab}.$

3. Add $a+b\sqrt{-c}$ and $a-b\sqrt{-c}$. *Ans. 2a.*
4. Add $a+b\sqrt[4]{-1}$ and $a-\sqrt[4]{-256}$. *Ans. (2a+b-4)\sqrt[4]{-1}.*
5. Add $ax+\sqrt{-by}$ and $by-\sqrt{-ax}$.
6. Add $\frac{2-\sqrt{-2}+\sqrt{-3}}{4}$, $\frac{\sqrt{3}+\sqrt{-18}+\sqrt{-75}}{2}$ and $-\frac{\sqrt{-8}-\sqrt{-12}}{3}$. *Ans. $\frac{1+\sqrt{3}}{2} + \frac{7\sqrt{-2}+41\sqrt{-3}}{12}$.*
7. Add $\frac{2-\sqrt{-12}}{4}$, $\frac{6-\sqrt{-8}}{4}$, $\frac{-3-\sqrt{-2}+\sqrt{-3}}{9}$ and $-\frac{2-\sqrt{-18}}{6}$.
8. Add $\frac{3-\sqrt{-4}}{5}$, $\frac{8-\sqrt{-10}}{9}$, $\frac{2-\sqrt{-12}-\sqrt{-8}}{10}$ and $-\frac{8-\sqrt{-12}-\sqrt{-10}-\sqrt{-14}}{12}$.
9. From $\frac{8-\sqrt{-5}}{2}$ subtract $\frac{3-\sqrt{-4}}{5}$.
10. From $(a+b)\sqrt{-1}+\sqrt{d}$ subtract $3\sqrt{-ab^2}-4\sqrt{-a^2b}$.

III. MULTIPLICATION.

326. The rule of (312), depending on a lemma in which the quantities are assumed real, requires some modification when applied to imaginaries.

Thus, by the old rule, $\sqrt[4]{-a} \cdot \sqrt[4]{-x} = \pm \sqrt[4]{ax}$; but this is not the true result, for:

$$\sqrt[4]{-a} = \sqrt[4]{a}\sqrt[4]{-1}; \sqrt[4]{-x} = \sqrt[4]{x}\sqrt[4]{-1}. \text{ Hence,}$$

$$\sqrt[4]{-a}\sqrt[4]{-x} = \sqrt[4]{a}\sqrt[4]{x}(\sqrt[4]{-1})^2 = \sqrt[4]{ax}\sqrt{-1}.$$

327. To Deduce a Rule for multiplying Imaginaries.

Let it be required to multiply $\sqrt[2n]{-a}$ by $\sqrt[2n]{-b}$.

$$\sqrt[2n]{-a} = \sqrt[2n]{a}\sqrt[2n]{-1}; \sqrt[2n]{-b} = \sqrt[2n]{b}\sqrt[2n]{-1}. \text{ Hence,}$$

$$\sqrt[2n]{-a} \cdot \sqrt[2n]{-b} = \sqrt[2n]{a}\sqrt[2n]{b}(\sqrt[2n]{-1})^2 = \sqrt[2n]{ab}\sqrt{-1}.$$

If there were s factors to be multiplied, on placing each in the form, $\sqrt[n]{x\sqrt[n]{-1}}$, and multiplying as in this case, the product would evidently be $\sqrt[n]{abcd \dots (\sqrt[n]{-1})^s}$.

If $2n$ is a multiple of s we may obtain $(\sqrt[n]{-1})^s$ by dividing the index of the radical by s .

Hence the following

RULE.—Place each radical in the form $\sqrt[n]{x\sqrt[n]{-1}}$. Multiply the real factors, (by the ordinary rules), and after their product write the required power of $\sqrt[n]{-1}$.

IV. INVOLUTION.

328. Corollary I.—If in the preceding, $abcd$, etc., are equal, we have $(\sqrt[n]{-a})^s = (\sqrt[n]{a})^s (\sqrt[n]{-1})^s$. Hence, for the **Involution** of imaginaries we have the following

RULE.—Put the given radical under the form $\sqrt[n]{x\sqrt[n]{-1}}$. Raise the real factor to the required power, by the ordinary rules; after the result write the required power of $\sqrt[n]{-1}$.

V. EVOLUTION.

329. Corollary II.—Conversely, since

$$(\sqrt[n]{-a})^s = (\sqrt[n]{a})^s (\sqrt[n]{-1})^s, \quad \sqrt[s]{(\sqrt[n]{a})^s (\sqrt[n]{-1})^s} = \sqrt[n]{-a}.$$

Hence, for **Evolution** of imaginaries, we have the following

RULE.—Separate the given quantity into two factors, one real, the other some power of $\sqrt[n]{-1}$. Extract the required root of each factor separately, and take the product of these roots for the required root.

EXAMPLES.

$$1. (-\sqrt[4]{-a})^{12} = ? \quad (-\sqrt[8]{-ab(x+y)})^7 = ? \quad (a + \sqrt{-b})(a - \sqrt{-b}) = ?$$

$$2. \text{ Multiply } \sqrt{\frac{x^3b^4}{x^4+2x^2b^4+b^4}} - \frac{x^4b^3}{x^4+2x^2b^3+b^4} \text{ by } \sqrt[4]{\frac{-3(x^4+b^4)}{x^3b^3}} - \frac{2(1+x^2b^2(x^4+b^4))}{x^3b^3}.$$

$$3. \left(\sqrt[n]{\frac{b}{-x}} + 1 - \frac{3\sqrt[n]{-x}}{b} \right) \cdot \left(\frac{2b}{\sqrt[n]{-x}} - 2 - \frac{6\sqrt[n]{-x}}{b} \right) = ?$$

$$\text{Ans. } -\frac{2b^2\sqrt[n]{-x}}{x} - 14 + \frac{18\sqrt[n]{-x}}{b^2}.$$

$$4. \left(2\sqrt[n]{-y} - \frac{b}{3} + 6c \right) \left(3\sqrt[n]{-y} + \frac{b}{2} - 6c \right) = ?$$

$$\text{Ans. } 6\sqrt[n]{-y} + 6c\sqrt[n]{-y} - \frac{b^2}{6} + 5bc - 36c^2.$$

$$5. \left(-\frac{2+\sqrt{3}}{2} + \frac{1+2\sqrt{-3}}{2} \right)^2 = ? \quad \left(-\frac{a+\sqrt{-3a^2}}{2} \right)^2 = ?$$

$$6. (\sqrt{-9})^2 = ? \quad (\sqrt{-9})^{12} = ? \quad (7\sqrt{-5})^2 = ? \quad (8\sqrt[n]{-7})^2 = ? \\ (\sqrt[n]{-8})^{12} = ? \quad (\sqrt[n]{-10})^2 = ? \quad (\sqrt[n]{-m})^r = ? \quad (\sqrt[n]{-82})^2 = ? \\ (a \pm \sqrt{-b})^2 = ? \quad (a + \sqrt{-b})^2 = ?$$

$$7. \sqrt{\sqrt{-\frac{1}{7}}} = ? \quad \sqrt{\sqrt[n]{-8}} = ? \quad \sqrt[3]{\sqrt[n]{-27a^3b^3c^3}} = ?$$

$$\sqrt[5]{\sqrt[10]{-ab(x+y)}} = ? \quad \left(\frac{a}{a+\sqrt[n]{-a}} \right)^4 = ?$$

VI. DIVISION.

330. To Deduce a Rule for the division of Imaginary quantities of any degree.

The rule of (314), depending on a proposition (217), equally true, whether the quantities are real or imaginary, is applicable without modification, when either the dividend or divisor is imaginary, and when both are imaginary.

Thus, if n is even,

$$\frac{\sqrt[n]{-x}}{\sqrt[n]{-y}} = \frac{\sqrt[n]{x}\sqrt[n]{-1}}{\sqrt[n]{y}\sqrt[n]{-1}} = \frac{\sqrt[n]{x}}{\sqrt[n]{y}} = \sqrt[n]{\frac{x}{y}} = \sqrt[n]{\frac{-x}{-y}};$$

and $\frac{\sqrt[n]{-x}}{\sqrt[n]{y}} = \sqrt[n]{-\frac{x}{y}}; \quad \frac{\sqrt[n]{x}}{\sqrt[n]{-y}} = \sqrt[n]{-\frac{x}{y}}.$

EXAMPLES.

1. Divide $2\sqrt{-8}+4\sqrt{-10}$ by $2\sqrt{-2}$.
2. Divide $8+12\sqrt[4]{-3}+6\sqrt{-3}+\sqrt[4]{-27}$ by $2+\sqrt[4]{-3}$.
Ans. $4+4\sqrt[4]{-3}+\sqrt{-3}$.
3. Divide $a^3-3a^2\sqrt[4]{-b}-b^2\sqrt[4]{-b^3}+3a\sqrt{-b}$ by $a+\sqrt[4]{-b}$.
4. Divide $14x^{\frac{2n}{3}}\sqrt[3]{-a}-8\sqrt[2n]{-a}+x^{\frac{2n}{3}}\sqrt[3]{-a}-7x^{\frac{2n}{3}}\sqrt[3]{-a}$ by $x\sqrt[6]{-a}-4\sqrt[6]{-a}$.
5. $2a^3\sqrt[4]{-x}-2a^2by\sqrt[4]{-x^3}+2ab^2y^3\sqrt[4]{-x}-2b^3y^4\sqrt[4]{-x}$ by $2a\sqrt[4]{-x}-2by\sqrt[4]{-x}$.

SECTION III.

IMPORTANT TRANSFORMATION.

331. To Transform a fraction with an irrational denominator into one of which the denominator is rational.

A. *The denominator has only a monomial irrational factor.*

Let $\frac{a}{b\sqrt[n]{c}}$ represent any fraction of which the denominator

has merely a monomial irrational factor, and suppose $\sqrt[n]{c}$ to be in its simplest form. Multiplying both terms of the given fraction by $\sqrt[n]{c^{n-1}}$ (148), we obtain $\frac{a\sqrt[n]{c^{n-1}}}{bc}$, a fraction of which the denominator is rational. As the letters employed may represent any quantities whatever, we deduce the following

RULE.—Reduce the given fraction to its simplest form, and multiply both terms by the $(n-1)$ th power of the irrational factor of the denominator.

Scholium.—From (157) we derive another method of effecting this transformation.

$$\frac{a}{b\sqrt[n]{c}} = \frac{a}{bc^{\frac{1}{n}}} = \frac{ac^{-\frac{1}{n}}}{b} = \frac{a\sqrt[n]{c^{-1}}}{b}.$$

B. *The irrational factor of the denominator is in the form of a binomial.*

The fraction will, in this case, be of the form $\frac{x}{\sqrt[n]{a} \pm \sqrt[n]{b}}$, x being positive or negative, entire or fractional, monomial or polynomial, etc.

Since $(\sqrt[n]{a})^n = a$ and $(\sqrt[n]{b})^n = b$, $a+b$ is exactly divisible by $\sqrt[n]{a} + \sqrt[n]{b}$ if n be odd, $a-b$ is so divisible if n be even, and $a-b$ is always exactly divisible by $\sqrt[n]{a} - \sqrt[n]{b}$ (96-9, 196).

$$\frac{a+b}{\sqrt[n]{a} + \sqrt[n]{b}} = \sqrt[n]{a^{n-1}} - \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} - \dots \pm \sqrt[n]{b^{n-1}} \quad (1)$$

$$\frac{a-b}{\sqrt[n]{a} - \sqrt[n]{b}} = \sqrt[n]{a^{n-1}} + \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} + \dots + \sqrt[n]{b^{n-1}} \quad (2)$$

If, then, $\sqrt[n]{a} + \sqrt[n]{b}$ be multiplied by the first quotient, the product will be $a+b$ or $a-b$, according as n is odd or even; if $\sqrt[n]{a} - \sqrt[n]{b}$ be multiplied by the second quotient, the product will be $a-b$. The multiplication here indicated will therefore enable us to get rid of the irrational factor of the denominator, and, that the value of the given fraction may not be altered, the numerator must be multiplied by the quantity by which the denominator is multiplied. Hence the

RULE.—*Multiply both terms of the given fraction, reduced to its lowest terms, by $\sqrt[n]{a^{n-1}} \pm \sqrt[n]{a^{n-2}b} + \sqrt[n]{a^{n-3}b^2} \pm \dots \pm \sqrt[n]{b^{n-1}}$, using the upper signs if the denominator is of the form $\sqrt[n]{a} - \sqrt[n]{b}$, and the signs of (1) if it is of the form $\sqrt[n]{a} + \sqrt[n]{b}$.*

CONVERSELY.—*If the irrational factor of the denominator be of the form $\sqrt[n]{a^{n-1}} \pm \sqrt[n]{a^{n-2}b} + \dots \pm \sqrt[n]{b^{n-1}}$, the required transformation may be effected by multiplying both terms of the fraction by $\sqrt[n]{a} \mp \sqrt[n]{b}$.*

Corollary.—*If $n=2$, the rationalising factor*

$$\sqrt[n]{a^{n-1}} \pm \sqrt[n]{a^{n-2}b} + \dots \text{ becomes } \sqrt{a} \pm \sqrt{b}.$$

Schollum.—*A factor may be found that will rationalize any binomial surd.*

C. The other forms of fractions with denominators containing irrational factors of the n th degree are not of sufficient importance to justify a detailed treatment of them here. The following forms, having irrational factors of the second degree, should, however, be carefully noted.

I. $\frac{x}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$; to rationalize the denominator multiply both terms of the fraction by $\sqrt{a}+\sqrt{b}-\sqrt{c}$. There results $\frac{x(\sqrt{a}+\sqrt{b}-\sqrt{c})}{a+b-c+2\sqrt{ab}}$; put $a+b-c=m$, and

$\frac{x(\sqrt{a}+\sqrt{b}-\sqrt{c})}{m+2\sqrt{ab}}$ may be treated as indicated in B.

II. $\frac{x}{\pm(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{d})} = \frac{x}{\pm(\sqrt{a}+\sqrt{b})\pm(\sqrt{c}+\sqrt{d})}$
 $= \frac{\pm x(\sqrt{a}+\sqrt{b}) \mp x(\sqrt{c}+\sqrt{d})}{a+b-c-d \pm (2\sqrt{ab}-2\sqrt{cd})}$. Let $a+b-c-d=n$; the denominator is then in form a trinomial, and the fraction may be treated by I.

Scholium.—The foregoing principles may be utilized in transforming a fraction having an irrational numerator into one with a rational numerator.

332. The transformations A, B, C, are used in finding the approximate values of fractions having radical terms. We could get approximately the values of such fractions by finding the approximate values of the different radicals, and performing the operations indicated. But, on completing the work, we should not be able to tell how far we might be from the truth. By the use of the foregoing transformations, however, we are enabled to determine the degree of approximation.

To ILLUSTRATE: take, 1°, $\frac{12\sqrt{8}}{\sqrt{10}}$ $12\sqrt{8}=\sqrt{1152}=34$; $\sqrt{10}=3$; (in each case to within 1). Hence the approximate value of the fraction is $\frac{34}{3}=11\frac{1}{3}$. Now $\frac{12\sqrt{8}}{\sqrt{10}}=\frac{12\sqrt{80}}{10}=\frac{\sqrt{11520}}{10}=\frac{107}{10}=10.7$. We cannot tell

how far the first result is from the truth, but we know the second to be true to within .1, the numerator being true to within 1.

2°. Take the fraction $\frac{6\sqrt{7}}{8\sqrt{2}+10\sqrt{3}}$ $\left\{ \begin{array}{l} 6\sqrt{7}=\sqrt{252}=16, \text{ to within } 1; \\ 8\sqrt{2}=\sqrt{128}=11, \text{ to within } 1; \\ 10\sqrt{3}=\sqrt{300}=17, \text{ to within } 1; \end{array} \right.$
 The value of this fraction, obtained by taking the approximate values of the radicals as they stand, is
 $\frac{16}{11+17} = \frac{16}{28} = \frac{8}{14}$. How far this may be from the truth we cannot tell.

$$\frac{6\sqrt{7}}{8\sqrt{2}+10\sqrt{3}} = \frac{6\sqrt{7}(8\sqrt{2}-10\sqrt{3})}{128-300} = \frac{48\sqrt{14}-60\sqrt{21}}{-172} =$$

$$\frac{\sqrt{32256}-\sqrt{75600}}{-172} = \frac{179-275}{-172} = \frac{96}{172}.$$

or, if we take the negative sign of the second radical, $\frac{454}{-172}$. We know these results to be true to within $\frac{1}{\pm 172}$; we take either result according as we take the positive or negative sign of the radicals (211).

EXAMPLES.

Transform the following fractions into equivalent ones with rational denominators:

$$1. \frac{ab\sqrt{2x}}{c\sqrt{4x^3}}; \frac{r\sqrt{x}}{\sqrt{a^3-x^3}}; \frac{\sqrt{x-a}}{\sqrt[3]{x^3-a^3}}; \frac{\sqrt[4]{x^3-a^3}}{\sqrt[4]{x^3+a^3}}; \frac{\sqrt[m]{a+x}}{\sqrt[m]{a-x}};$$

$$\frac{\sqrt[m]{1+x^3}}{\sqrt[n]{1+x^3}}; \frac{1}{\sqrt{3}}; \frac{2}{\sqrt[3]{4}}; \frac{3}{4\sqrt[5]{5}}; \frac{\sqrt[5]{7}}{\sqrt[6]{8}}; \frac{\sqrt[9]{12}}{\sqrt[10]{9}};$$

$$\frac{1+2\sqrt{2}}{\sqrt[6]{7}}; \frac{\sqrt[5]{6}}{\sqrt[3]{7}}; \frac{5\sqrt[2]{2}}{6\sqrt[3]{3}}.$$

$$2. \frac{a}{\sqrt{a+b}}; \frac{x}{y-\sqrt{m}}; \frac{a+x}{\sqrt{a}\pm\sqrt{x}}; \frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}};$$

$$\frac{\sqrt[3]{x}+\sqrt[3]{y}}{\sqrt[3]{x}-\sqrt[3]{y}}; \frac{(a+x)^3}{\sqrt[5]{a+x}-\sqrt[5]{y}}.$$

$$3. \frac{\sqrt{a+x}-\sqrt{a-x}}{\sqrt{a+x}+\sqrt{a-x}}; \frac{\sqrt{x+y+z}-\sqrt{x-y-z}}{\sqrt{x+y+z}+\sqrt{x-y-z}}.$$

$$\text{Ans. } \frac{a-\sqrt{a^2-x^2}}{x}; \frac{x-\sqrt{x^2-y^2-z^2-2yz}}{y+z}.$$

$$4. \frac{ax+byz-\sqrt{a^2x^2+b^2y^2z^2}}{ax-byz+\sqrt{a^2x^2+b^2y^2z^2}}; \frac{\sqrt{1+x^2+2xy+y^2}-x-y}{\sqrt{1+x^2+2xy+y^2}+x+y}.$$

$$\text{Ans. } \frac{\sqrt{a^2x^2+b^2y^2z^2}-ax}{byz};$$

$$1+2x^2+4xy+2y^2-2(x+y)\sqrt{1+x^2+2xy+y^2}.$$

$$5. \frac{\sqrt{\frac{1}{2}}+\sqrt{\frac{1}{6}}}{\sqrt{\frac{2}{3}}+\sqrt{\frac{1}{6}}}; \quad \frac{\sqrt{2}+\sqrt{3}}{3\sqrt{2}+2\sqrt{3}}; \quad \frac{a-b}{\sqrt{a}+\sqrt{b}};$$

$$\frac{4}{2-\sqrt{2}+\sqrt{6}}; \quad \frac{8}{\sqrt[3]{5}+\sqrt[3]{3}}; \quad \frac{5}{2-\sqrt[3]{3}}.$$

Ans.

$$\frac{1+\sqrt{3}}{3}; \quad \frac{1}{6}\sqrt{6}; \quad \sqrt[3]{a^3}-\sqrt[3]{a^2b}+\sqrt[3]{ab^2}-\sqrt[3]{a^2b^2}+\sqrt[3]{ab^3}-\sqrt[3]{b^3};$$

$$1-\sqrt{2}+\sqrt{3}; \quad \sqrt[3]{25}-\sqrt[3]{15}+\sqrt[3]{9}; \quad 4+2\sqrt[3]{3}+\sqrt[3]{9}.$$

$$6. \frac{\sqrt{2}-2\sqrt{3}+\sqrt{6}}{1+\sqrt{2}-\sqrt{3}}; \quad \frac{2-2\sqrt{2}-2\sqrt{3}-2\sqrt{6}}{5+5\sqrt{2}+5\sqrt{3}+5\sqrt{6}};$$

$$\frac{2+\sqrt{6}}{8-16\sqrt{2}-8\sqrt{3}+8\sqrt{6}}.$$

$$\text{Ans. } 2-\sqrt{2}; \quad \frac{2}{5}(\sqrt{6}-\sqrt{2}-\sqrt{3})$$

$$7. \frac{1+\sqrt{2}+\sqrt{3}+\sqrt{5}}{\sqrt{6}+\sqrt{7}-\sqrt{8}+\sqrt{10}}; \quad \frac{7}{\sqrt[4]{4}-\sqrt[4]{-3}}; \quad \frac{10}{\sqrt[4]{-3}-\sqrt[4]{-13}}.$$

$$8. \frac{v}{\sqrt[m]{a^c}-\sqrt[r]{b^c}} = \frac{v}{a^{\frac{c}{m}}-b^{\frac{c}{r}}} = \frac{v}{x^c-y^c} \text{ (placing } a^{\frac{1}{m}}=x \text{ and } b^{\frac{1}{r}}=y)$$

Let p be the lowest common multiple of m and r ; then $x^p-y^p = a^{\frac{p}{m}}-b^{\frac{p}{r}}$ is rational. $x^p=(x^c)^p$ and $y^p=(y^c)^p$. Hence, (96) $\frac{x^p-y^p}{x^c-y^c} = (x^c)^{p-1}+(x^c)^{p-2}y^c+\dots+(y^c)^{p-1}(1)$.

If, then, both terms of $\frac{v}{x^c-y^c}$ be multiplied by the quotient

(1), and x^c and y^c be replaced by their equals, $a^{\frac{c}{m}}$ and $b^{\frac{c}{r}}$, the denominator will be rational.

CHAPTER VII.

EQUATIONS.

SECTION I.

EQUATIONS OF THE FIRST DEGREE.

333. An Equation is an algebraic expression of the equality of two quantities.

The **Members** of an equation are the divisions made by the sign of equality. The part preceding the sign is the **First Member**, and the part following the sign is the **Second Member**.

An equation usually involves both *known* and *unknown* quantities; the former being represented by *figures* or the *leading* letters of the alphabet, and the latter by the *closing* letters.

334. A Numerical Equation is one in which the known quantities are all represented by figures.

ILLUSTRATION. $3x^2 - 7 + 3y = 26$ is a numerical equation.

335. A Literal Equation is one in which the known quantities are represented, either partly or exclusively, by letters.

ILLUSTRATION.
$$\left. \begin{array}{l} ax^2 + bx + cy + d = g \\ 3ax + 2by + 6 = a^2 - b \end{array} \right\} \text{ are literal equations.}$$

336. An equation is in its **Simplest Form** when the second member is 0, and the first member is in its simplest form.

ILLUSTRATION. (a)
$$\left. \begin{array}{l} 3x^2 + 2x + 7 + x = x^2 - 5 \\ bx + cy + gh + 7 = rx + y \end{array} \right\} \text{ are not in the simplest form.}$$

(b)
$$\left. \begin{array}{l} 4x^2 + 3x + 12 = 0 \\ (b-r)x + (c-1)y + (gh+7) = 0 \\ ax + by + c = 0 \end{array} \right\} \text{ are in the simplest form.}$$

337. The Absolute Term of an equation is the term that is absolutely or completely known; it is the algebraic sum of the *known terms* of the equation, and is usually the last term of the first member if the equation is in its simplest form.

ILLUSTRATION. 12 , $(gh+7)$, and c are the absolute terms of the equations in (336, b).

338. The Degree of an equation is the greatest number of unknown factors in any term. It is denoted by the greatest sum of the exponents of the unknown quantities in any term, the exponent 1 being understood where no other is expressed. If, then, but one unknown quantity enters the equation, the degree is denoted by the exponent of the highest power of this quantity in any term.

339. An equation is *satisfied* when, certain quantities being substituted for the unknown quantities, the first member of the equation is the same as the second. The quantities substituted are called *Values of the unknown quantities*.

340. A Root of an equation is a quantity, by the substitution of which the equation is satisfied.

341. The Transformation or Reduction of an equation is the process of changing its form without destroying the equality of the members.

We cannot speak of the *value of an equation*. An equation is simply a proposition written with algebraic characters, and the transformation of the equation is merely the process of modifying the *method* of statement without destroying the *truth* of the proposition.

342. The Solution of an equation is the process of finding a root of the equation.

An equation with but one unknown quantity may have several roots, thus, $x^2-7x+10=0$ will be satisfied by the substitution of either 2 or 5 for x .

343. The Verification of an equation is the process of substituting for the unknown quantities their values, and

performing the operations indicated, thus making the first member identical with the second.

344. Classification of Equations.—Equations are classified:

1°. According to the *number of unknown quantities*, into Equations with *one* unknown quantity, and Equations with *two* or *more* unknown quantities.

2°. According to the *degree*, into

Equations of the **First degree**, as $\begin{cases} 3x+7=29+x. \\ ax+by=c+f. \end{cases}$

Equations of the **Second degree**, as $\begin{cases} x^2+2x=7. \\ 3x^2-25=4x+2x^2+1. \\ x^2+y^2+z^2=m^2+u^2+v^2. \end{cases}$

And so on to

Equations of the **nth degree**, as $\begin{cases} x^n+Px^{n-1}+Qx^{n-2}=75. \\ x^n+y^n+z^n=54-b. \\ x^n+y+76=r^{n-1}+s. \end{cases}$

Before deciding on the *degree* of an equation, the equation must be so transformed that it may be free from radicals.

3°. According to the *number of ways in which they may be satisfied*.

I. Equations satisfied by only a *limited* number of values of the unknown quantities. These equations are generally called **Common Equations**.

II. Equations satisfied by an *unlimited* number of values of the unknown quantities. These are:

1°. *Identical*; or, 2°. *Indeterminate*.

An Identical Equation is an equation in which the two members are precisely alike, or one member is the result of operations indicated in the other; and the equation is satisfied by the substitution of *any quantities whatever* for the unknown quantities.

ILLUSTRATION. $x=x$; $(x+y)^2=x^2+2xy+y^2$; $\frac{x^2-y^2}{x-y}=x^2+xy+y^2$.

An Indeterminate Equation is an equation (containing more than one unknown quantity) that may be satisfied by

an infinite number of *sets of values* of the unknown quantities.

An indeterminate equation is not satisfied by the substitution of any values whatever for the unknown quantities, but yet it is satisfied in an infinite number of ways. Thus the equation $x-y=2$ will not be satisfied by the substitution of any quantities whatever for x and y , but will be satisfied by the substitution of any number whatever for y , and a number greater by 2 for x .

4°. According to the *nature of the relations between the known and unknown quantities.*

I. Algebraic Equations are those equations wherein the relations of the quantities can all be expressed by addition, subtraction, multiplication, division, involution, the exponents being known, and evolution, the indices being known.

ILLUSTRATION. $3x^2-2mx+7x=3$; $5x-2y+w=7$.

II. Transcendental Equations are those that are not algebraic.

ILLUSTRATION. $3x^x=7$; $5a^x=13^{x+y}$; $7^x=7^x$; $\log a=b$.

The principal transcendental equations that will be treated in this book are *Exponential* equations.

An Exponential Equation is an equation in which any of the exponents are unknown quantities.

ILLUSTRATION. $8x=a^{m+y}$; $b^x=c^y$; $a^{x+a}=b^{x+a}$; $x^y=x^y$.

5°. According to the *form* of the exponents:

I. Rational Equations are those in which the exponents of the unknown quantities are all entire.

II. Radical Equations are those in which any of these exponents are fractional, or any of the unknown quantities are affected by the radical sign: as $\sqrt{x}=3b+x^{\frac{2}{3}}$; $\sqrt[3]{x+a}=7$.

These equations must be so transformed as to be free of radicals before the degree of the equation can be determined.

The classes just given evidently pass into one another, so that an equation may be at the same time in two or more classes, provided these classes be of different kinds. Thus, $3^x=3^x$ is an identical and exponential equation, but it is evident that an equation can not be at the same time *Rational* and *Radical*.

6°. According to the nature of the coefficients into

I. Numerical.

II. Literal.

345. From the definition of an equation it follows that:

I. *The two members must be composed of quantities of the same kind.*

II. *The two members must be numerically equal.*

III. *The two members must have the same essential sign.*

346. The following axiom is used in transforming and solving equations.

If the members of an equation be subjected to the same operation, the equality will not be destroyed.

TRANSFORMATIONS.

347. I.—To Deduce a Rule for reducing an equation in which some terms are fractional to an equation in which all of the terms are entire.

Let us assume the equation $\frac{a}{b}x + \frac{c}{d}x - 25 = \frac{x}{m} + r$ (1). If now, the numerator of each fraction were exactly divisible by the denominator, all the terms in the equation would be entire. Hence, if we can multiply each member of the equation by such a quantity as will render this division possible, the equation will be in the desired form when the indicated operations are performed. If we multiply both members by the denominator of any fraction we shall make the term involving that fraction entire, but not necessarily any of the other terms. If we multiply both members by a common multiple of the denominators of two of the fractions we shall make the terms involving these two fractions entire, but not necessarily the terms involving the other fractions. Multiplying both members by a common multiple of the denominators, however, renders all terms involving fractions entire when the indicated operations are performed. For each numerator will contain, as a factor, a

multiple of the corresponding denominator, and must therefore be exactly divisible by that denominator. As we must multiply by a common multiple to render the *terms entire*, if we multiply by the *lowest* common multiple, the resulting terms will be the *lowest* entire terms possible.

Thus if we multiply both members of (1) by b , the first term resulting will reduce to an entire term, but not so the others. If we multiply by bd , we obtain $adx+bcx-25bd=\frac{bd}{m}x+bdr$, in which we still have the fractional term $\frac{bd}{m}x$. Finally, if we multiply by bdm , the lowest common multiple of all the denominators, we obtain $adm+bcmx-25bdm=bdx+bdmr$, an equation of the required form.

As we could reason in a similar manner on *any equation whatever*, we deduce the following

RULE.—*Multiply both members of the given equation by the lowest common multiple of the denominators, reducing, in the result, fractional to entire terms.*

348. Scholium I.—It is best to reduce fractional to entire terms mentally, as the multiplication proceeds, writing only the results, in each case.

349. This transformation is called *Clearing the Equation of Fractions*.

EXAMPLES.

Clear the following equations of fractions:

$$1. \frac{3x}{7} - \frac{2x}{3} + \frac{x}{6} = 10. \quad \text{Ans. } 18x - 28x + 7x = 420.$$

$$2. \frac{7x-3}{4} + \frac{x-5}{3} - \frac{2x-8}{5} = 7. \\ \text{Ans. } 105x - 45 + 20x - 100 - 24x + 96 = 420.$$

A minus sign prefixed to the vinculum of a fraction affects *every term* of the numerator.

$$3. \frac{x-1}{2} + \frac{x-2}{3} - \frac{x-3}{4} + s = \frac{2x-7}{6}. \\ \text{Ans. } 6x - 6 + 4x - 8 - 3x + 9 + 12s = 4x - 14.$$

$$4. \frac{a}{b}x - \frac{c}{d}x - \frac{e-f}{m}x = l + r.$$

$$\text{Ans. } admx - bdmx - ebdx + fbdx = bdlm + bdmr.$$

$$5. \frac{x}{a-b} + \frac{3-x}{a^2-b^2} + 7 = \frac{x-1}{a+b} - \frac{x+5}{a^2c-b^2c}.$$

$$\text{Ans. } acx + bcx + 3c - cx + 7a^2c - 7b^2c = acx - bcx - ac + bc - x - 5$$

350. II.—To Deduce a Rule for transposing a term from one member of an equation to the other.

Let it be required to transfer all the known terms of the equation $rx + cx + m = n - fx$ (1) to the second member, and all the other terms to the first.

If, to both members of (1) we add $fx - m$, we shall not destroy the equality of the members. Adding, we obtain $rx + cx + m + fx - m = n - fx + fx - m$ (2), or, since $m - m = 0$, and $-fx + fx = 0$, $rx + cx + fx = n - m$ (3).

Comparing (3) with (1), we see that the required transfer has been effected, the signs of the transferred terms being changed. As we may treat any equation whatever in a similar manner, we deduce the following

RULE.—*Drop the term to be transferred from the member in which it stands, and place it, with its sign changed, in the other member.*

351. This transformation is called *Transposition*.

352. Corollary I.—*Changing the sign of every term of an equation does not affect the equality of the members.*

353. Corollary II.—*If any term of the first member be algebraically equal to any term of the second, this term may be expunged from both members of the equation.*

EXAMPLES.

Transform the following equations so that all the known terms may be in the second members, and all the other terms in the first members.

$$1. 7x-2b+x-(b-c+5)=3x+4.$$

$$\text{Ans. } 7x+x-3x=2b+b-c+5+4.$$

$$2. (a-b)x-(3-b)c=7x+(5-d)x.$$

$$\text{Ans. } (a-b)x-7x-(5-d)x=(3-b)c.$$

$$3. (x-1)a+(a-1)x=2x-3+a.$$

$$\text{Ans. } ax+(a-1)x-2x=a-3+a.$$

$$4. 5y-3a+bcy=8+a-y. \quad \text{Ans. } 5y+bcy+y=8+a+3a.$$

$$5. ax+bx+g+f+3x=7-x+dx+g.$$

$$\text{Ans. } ax+bx+3x+x-dx=7-f.$$

354. III.—To Deduce a Rule for transforming an equation containing fractional exponents into another in which all the exponents are entire.

Let us take the equation $ax^{\frac{1}{2}}+3tx^{\frac{3}{5}}+27x^{\frac{1}{3}}=10x^{\frac{1}{6}}+c$, and transform it so that the new equation may have no fractional exponents. We have already (181) found that we may multiply both terms of any fractional exponent by the same quantity without affecting the result of the process indicated by the exponent. Hence, the value of the members of the equation will not be altered if we transform each fractional exponent into a fraction with the denominator 6; doing so, we obtain $ax^{\frac{3}{6}}+3tx^{\frac{4}{6}}+27x^{\frac{2}{6}}=10x^{\frac{1}{6}}+c$. Representing $x^{\frac{1}{6}}$ by y , we have $ay^3+3ty^4+27y^2=10y+c$. We have not at any step affected the equality of the members, and we have obtained an equation free from fractional exponents. If the value of x be required, we may, on finding the value of y , substitute for x the value y^6 . (For, since $x^{\frac{1}{6}}=y$, raising both equals to the 6th power, we have $x^1=y^6$).

As we could treat in a similar manner any equation in which are fractional exponents, we deduce the following

RULE.—Reduce the fractional exponents to equivalent fractions having the lowest common denominator. Substitute for the old unknown quantity a new one raised to the power indicated by this common denominator.

355. Scholium.—By transforming radicals into quantities with fractional exponents, we may apply this rule to clearing radical equations of radicals.

356. Scholium II.—We treat the exponents of the *unknown* quantities only, since the roots and powers of *known* quantities are known, exactly or approximately.

EXAMPLES.

Transform the following equations in the manner indicated:

$$1. mx^{\frac{1}{2}} + 2cx^{\frac{1}{3}} + cd = rx^{\frac{1}{6}}. \quad \text{Ans. } my^2 + 2cy^3 + cd = ry^6.$$

$$2. 4\sqrt{x-2} + 7x^{\frac{1}{4}} + l = a + b\sqrt{x}. \quad \text{Ans. } 4y^2 - 2 + 7y + l = a + by^2.$$

$$3. 5cx^{\frac{1}{3}} - 2d\sqrt{x} + 7 = 3\sqrt{x} + cm.$$

$$4. \sqrt[5]{y} + 3\sqrt{y} + dy^{\frac{1}{6}} = 4aby^{\frac{1}{3}} + ly^{\frac{1}{2}}.$$

$$5. (a-1)\sqrt{x} - (b+1)\sqrt[3]{x} + c = 3m - \sqrt{x}.$$

EQUATIONS OF THE FIRST DEGREE.

I. EQUATIONS CONTAINING ONLY ONE UNKNOWN QUANTITY.

357. To Deduce a Rule for the solution of equations of the first degree containing but one unknown quantity.

An equation is solved when so transformed that the unknown quantity stands alone as one member, and a quantity completely known as the other. For, if this completely known quantity were substituted for the unknown quantity in the other member, the two members would be identical.

Let it be required to solve the equation,

$$\frac{a}{b}x - \frac{cx+g}{2d} = c - \frac{x}{d} + m \dots \dots \dots (1).$$

Clearing of fractions we obtain,

$$2adx - bcx - bg = 2bcd - 2bx + 2bmd \dots \dots \dots (2).$$

Transposing all terms containing the unknown quantity to the first member, and all the others to the second, we have

$$2adx - bcx + 2bx = bg + 2bcd + 2bmd \dots\dots\dots (3)$$

$$\text{or } 2(ad+b)x - bcx = bg + 2(c+m)bd \dots\dots\dots (4)$$

$$[2(ad+b) - bc]x = bg + 2(c+m)bd \dots\dots\dots (5)$$

If, now, we divide both members by $2(ad+b) - bc$, x will stand alone in the first member, and the second will be completely known. Dividing, we have, $x = \frac{bg + 2(c+m)bd}{2(ad+b) - bc}$.

Since the value of the unknown quantity in any other equation of the first degree may be found in a manner precisely similar, we deduce the following

RULE.—*Clear the given equation of fractions and perform the operations indicated.*

Transpose all terms containing the unknown quantity to one member, and all known terms to the other. Reduce each member to its simplest form.

Resolve the first member into two factors, viz: the unknown quantity and the algebraic sum of its coefficients, which sum is the coefficient of the unknown quantity.

Divide both members by the coefficient of the unknown quantity.

To Verify the value thus found:—*Substitute this value for the unknown quantity in the given equation and perform all the operations indicated. The members of the resulting equation will be identical.*

EXAMPLES.

Solve the following equations.

$$1. \quad \frac{2x}{5} - 7 + \frac{3x-5}{2} = 12 - \frac{x}{4}. \quad \text{Clearing of fractions,}$$

$$8x - 140 + 30x - 50 = 240 - 5x, \quad \text{Transposing,}$$

$$8x + 30x + 5x = 140 + 240 + 50. \quad \text{Reducing,}$$

$$43x = 430$$

$$x = 10$$

To verify this value,

$$\frac{20}{5} - 7 + \frac{30-5}{2} = 12 - \frac{10}{4}; \text{ or, } 4 - 7 + \frac{25}{2} = 12 - \frac{5}{2}; \text{ or,}$$

$$4 - 7 + \frac{30}{2} = 12; \text{ or, } 4 - 7 + 15 = 12; \text{ or, } 19 - 7 = 12; \text{ or, } 12 = 12.$$

$$2. \frac{2x}{3} + \frac{7x}{21} - x = 36 - x + \frac{5x}{3}. \quad x = -54.$$

$$3. \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 2x + 5 - \frac{x}{9} - \frac{15}{32}. \quad x = -5\frac{5}{8}.$$

$$4. 2x + 50 + (9-x)12 = 4x - (3-5x)6. \quad x = 4.$$

$$5. \frac{1}{2}x + \frac{1}{4}x + \frac{1}{3}x - \frac{1}{6}x - 72 = 24 - \frac{x}{12}. \quad x = 96.$$

$$6. \frac{1}{6} + \frac{x-5}{4} - \frac{x-4}{3} - \frac{x-3}{2} = \frac{x-1}{2} - \frac{x-2}{3} + \frac{10}{12}. \quad x = 1.$$

$$7. \frac{x-1}{2} + \frac{x-2}{3} + \frac{x-3}{4} = 27 - \frac{2x-5}{6}. \quad x = 21.$$

$$8. 5x - 15 + \frac{2x+4}{11} - \frac{6}{11} = 7\frac{2}{11} - \frac{8x-4}{2} - 3x. \quad x = 2.$$

$$9. \frac{8-3x}{7} - \frac{x-5}{3} = -2x - \frac{3x+11}{6} + \frac{49}{42}. \quad x = -2.$$

$$10. 17x - 23 + 15(x-2) = 54\frac{1}{2} - 17\frac{1}{2}x - \frac{17}{2}. \quad x = 2.$$

11.

$$\frac{5-x}{6} - 22 - \frac{x+8}{3} + 11x + \frac{5}{12} = \frac{13-5x}{12} + \frac{x-4}{2} + \frac{x-3}{6} + 19$$

$$x = 4.$$

$$12. \frac{x}{a} - \frac{x}{b} = \frac{4x}{b} - \frac{x}{a} + 8. \quad x = \frac{8ab}{2b-5a}.$$

$$13. \frac{x}{1-a} + \frac{x}{1+a} = 2. \quad x = 1 - a^2.$$

$$14. \frac{x}{a-b} + \frac{x}{a+b} = 1 - x. \quad x = \frac{a^2 - b^2}{a^2 + 2a - b^2}.$$

$$15. a + \frac{ax}{b} + 1 = \frac{cx}{d} - 1 + c. \quad x = \frac{bd(c-a-2)}{ad-bc}.$$

$$16. \frac{1+x}{2} - \frac{1+x}{3} = -\frac{1+a}{4} - \frac{1-a}{6}. \quad x = -\frac{a+7}{2}.$$

$$17. (2a-b)x - 4b - (b-5a)x = 3a - bx - 2a.$$

$$18. \frac{x}{a} - \frac{2x-1}{a-b} = \frac{x-2}{a-b} + \frac{b}{a}. \quad x = \frac{b(b-a)+3a}{2a+b}.$$

$$19. \frac{3c^2-a}{4} + \frac{2x-6}{5} = \frac{c^2}{3} - \frac{x+2}{2}.$$

$$20. (ab+c)x - 2(b+a)^2 = (a-c)x + b^2 - a^2. \quad x = \frac{3b^2+4ab+a^2}{ab+2c-a}.$$

$$21. \frac{2x-1}{3} - \frac{8x-4}{12} + \frac{3x}{8} = \frac{7x+4}{10} - \frac{4x+4}{9} + 1. \quad x=8.$$

$$22. \frac{7x+1}{5} - \frac{2x+3}{3} + \frac{2}{3} = \frac{12x+1}{5} - \frac{2x+34}{8} - 6. \quad x=7.$$

$$23. \frac{5x}{7} - \left(\frac{x}{2} - \frac{3x+2}{22} \right) = \frac{x}{14} + \frac{5x+7}{11} - 3. \quad x=14.$$

It is sometimes convenient to clear an equation of some (usually the simplest), of the denominators, and to effect whatever reductions may be expedient before disposing of other denominators. By pursuing this course we may be enabled to avoid troublesome operations and undesirable forms of equations. Take the equation,

$$\frac{10\frac{5}{8}x}{12} + \frac{8x+3a}{32} - \frac{x+2a}{5x-13} = \frac{9\frac{1}{12}x}{8}.$$

If we attempt to apply the rule at once we shall obtain terms involving x^2 , but we may avoid such terms by proceeding as just indicated. Thus, multiplying both members by 8, we obtain

$$\frac{85x}{12} + \frac{8x+3a}{4} - \frac{8(x+2a)}{5x-13} = 9\frac{1}{12}x,$$

$$\text{Or, } 7x + \frac{x}{12} + \frac{8x+3a}{4} - \frac{8(x+2a)}{5x-13} = 9\frac{1}{12}x.$$

Subtracting $7x$ from each member, we obtain

$$\frac{x}{12} + \frac{8x+3a}{4} - \frac{8(x+2a)}{5x-13} = 2\frac{1}{12}x.$$

Multiplying both members by 12, we obtain,

$$x+24x+9a-\frac{96(x+2a)}{5x-13}=25x;$$

Or, subtracting $25x$ from each member, and transposing,

$$\frac{96(x+2a)}{5x-13}, 9a=\frac{96(x+2a)}{5x-13}; \text{ whence}$$

$$45ax-117a=96x+192a,$$

$$45ax-96x=309a; (45a-96)x=309a$$

$$\therefore x=\frac{309a}{45a-96}.$$

$$24. \frac{16x-19}{27} + 3\frac{11x+18}{12x+8} = 5\frac{1}{3}x + \frac{2}{3} \dots\dots \text{Multiplying both}$$

members by 9, we have, $\frac{16x-19}{3} + 27\frac{11x+18}{12x+8} = 5\frac{1}{3}x + \frac{2}{3},$

$$\text{or} \quad 5\frac{1}{3}x - 6\frac{1}{3} + 27\frac{11x+18}{12x+8} = 5\frac{1}{3}x + \frac{2}{3}.$$

Subtracting $5\frac{1}{3}x$ from each member, and transposing,

$$27\frac{11x+18}{12x+8}=7; \text{ whence}$$

$$297x+486=84x+56; 213x=-430$$

$$x=2.02 \text{ (nearly)}$$

$$25. \frac{21x+3}{12} + \frac{27x-3}{8x+4} = \frac{7x+13}{4}. \quad x=5.$$

$$26. \frac{9x-12}{14} - \frac{7x-8}{5x-13} = \frac{36x-160}{56}. \quad x=6.$$

$$27. \frac{2x-1}{5} + \frac{3x+11}{10} - \frac{2x+17}{11} = \frac{3x+1}{4} - \frac{7x-41}{25} - \frac{21}{11}. \quad x=13.$$

$$28. \frac{12x+11}{5} - \frac{6x+7}{7} + \frac{x-1}{3} = \frac{8x+7}{9} + \frac{10x+23}{81} + 2. \quad x=7.$$

$$29. \frac{5x-7}{19} + \frac{3x+7}{17} - \frac{2x+9}{3} = \frac{7x+2}{13} - \frac{8x+28}{10}. \quad x=9.$$

$$30. \frac{7x+13}{38} - \frac{12x-88}{10} = \frac{x+15}{2} - \frac{2x+3}{7} - x. \quad x=9.$$

$$31. \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{5x+1}{11} - \left(\frac{x}{8} - \frac{3x+4}{38} - 16 \right). \quad x=24.$$

$$32. \frac{5x-4}{6} + \frac{3x-17}{7} - \frac{4x+3}{5} = \frac{17-2x}{8} - \frac{7x-41}{45}. \quad x=8.$$

$$33. \frac{9x-8}{2x-4} - \frac{27x-19}{6x-11} = 0. \text{ Clearing of fractions, we have} \\ 54x^2 - 147x + 88 - 54x^2 + 146x - 76 = 0, \text{ or } -x + 12 = 0, \\ \text{whence } x=12.$$

$$34. \frac{x-1}{x-3} - \frac{x-3}{x-5} = \frac{x-5}{x-7} - \frac{x-7}{x-9}. \quad x=6.$$

$$35. \left(x - \frac{1}{2} \right) \left(x + \frac{1}{2} \right) - (x-2)(x-4) = 17. \quad x = \frac{101}{24}.$$

$$36. \frac{x-a}{b} + \frac{x-b}{a} + (a+b)x = a^2 + b^2 \quad x = \frac{a^2 + b^2}{a+b}.$$

$$37. \frac{x-a}{bc} + \frac{x-b}{ac} + \frac{x-c}{ab} + \frac{x}{a^2+b^2+c^2} = \frac{1}{a+b+c}. \\ x = \frac{a^2+b^2+c^2}{a+b+c}.$$

$$38. \frac{x+b}{a+b} - \frac{x-b}{a-b} = \frac{ax+b^2}{a^2-b^2}. \quad x=b \frac{2a-b}{a+2b}.$$

$$39. \frac{x+a}{a-b} - \frac{x-a}{a+b} - \frac{bx-b^2}{a^2-b^2} = 0.$$

$$40. \frac{x-a}{2(a+b)} + \frac{x-2a+3b}{6b} = \frac{x-3b}{3(a-b)} + \frac{x-b}{4a}. \quad x=2a+b.$$

$$41. \frac{2bx+cx}{a} - \frac{ax+b-c}{b+c} + \frac{x}{a} = \frac{a^2+b-(b+c)^2+c}{b+c} + \frac{ab+x}{a}$$

$$42. \frac{x+a}{b} + \frac{x+b}{c} + \frac{x+c}{d} = \frac{d}{e}. \quad x = \frac{bc(d^2-ec)-de(b^2+ac)}{e(cd+bd+bc)}.$$

$$43. \frac{4ax}{2ab-2b^2} - \left(\frac{x+2a}{a-b} + \frac{x-2b}{2(a-b)} + \frac{3(x-a)}{b} \right) = 0.$$

$$44. \frac{x-b^4}{(a^2+b^2)^2} - \frac{x-a^2b^2}{a^4+b^4} = \frac{a^4-x}{(a^2+b^2)^2}. \quad x=a^4+a^2b^2+b^4.$$

$$45. \frac{x-b^2}{(a+b)^2} - \frac{x-a^2}{(a-b)^2} = \frac{a^4-b^4}{(a^2-b^2)^2}. \quad x = \frac{b^2+a^2}{2}.$$

$$46. \frac{1}{a} \left(x - \frac{b}{c} \right) - \frac{1}{c} \left(x - \frac{b}{d} \right) + \frac{1}{d} \left(x - \frac{b}{e} \right) = 0. \quad x = \frac{abc - abe + bde}{dce + ace - ade}$$

$$47. \left(x + \frac{a}{b} \right) \left(x - \frac{c}{b} \right) - (x-a)(x-b) = ax - \frac{cx}{b}.$$

$$48. (x-a)^2 + (x-b)^2 = 2(x+a)(x+b).$$

$$49. (x-a)^2 + (x-b)^2 + (x-c)^2 = 3(x-a)(x-b)(x-c).$$

$$x = \frac{a+b+c}{3}$$

$$50. \frac{b-a}{x-b} - \frac{a-2b}{x-b} + \frac{3x(a-b)}{x^2-b^2} + 1 = \frac{x^2}{x^2-b^2}.$$

$$51. \frac{a+x}{2(a^2-b^2)} + \frac{b+x}{3(a-b)} = \frac{2b+x}{a^2+ab} + \frac{1}{3}. \quad x=a-2b.$$

RADICAL EQUATIONS OF THE FIRST DEGREE.

358. When these equations have been cleared of radicals, they present no special difficulty, and are, in general, solved by the ordinary rule.

While no general rule for clearing an equation of radicals can be given, this process usually consists in so transforming the given equation, that the radicals stand alone in one member, performing such involution as will ensure the disappearance of one or more of the radicals, and repeating this until all the radicals have disappeared. Special methods of facilitating the clearing of an equation of radicals will be illustrated, in due course, by the solution of

appropriate examples. To attain skill in effecting this transformation, the student must, however, rely to a great extent on intelligent and extensive practice, based on habits of close and accurate observation.

EXAMPLES.

1. $\sqrt{3+\sqrt{6+\sqrt{x+4}}}=2$. Squaring both members,

$$3+\sqrt{6+\sqrt{x+4}}=4, \text{ or } \sqrt{6+\sqrt{x+4}}=1;$$

squaring both members of this equation, $6+\sqrt{x+4}=1$, or $\sqrt{x+4}=-5$; whence, by squaring again, $x+4=25$; whence $x=21$.

2. $\sqrt{15+\sqrt{2+\sqrt{2x-7}}}=4$. $x=4$.

3. $\sqrt{35+\sqrt[3]{3+\sqrt{x-9}}}=6$. $x=25$.

4. $\sqrt{3x+7}+5=12$. $x=14$.

5. $2\sqrt{6x+1}-8=2$. $x=4$.

6. $\sqrt[3]{11x+5}+4=7$. $x=2$.

7. $-2\sqrt[5]{3x+13}-27=-29$. $x=-4$.

8. $2\sqrt[9]{5x+\frac{89}{64}}+6=9$. $x=2$.

9. $\frac{\sqrt{x}-2}{\sqrt{x}-8}=\frac{\sqrt{x}-6}{\sqrt{x}}$. $x=16$.

10. $3\sqrt{x}-50+7\sqrt{x}=3\left(2-6\frac{2}{3}\sqrt{x}\right)-22+18\frac{2}{3}\sqrt{x}$. $x=9$.

11. $\frac{\sqrt{x}+4}{\sqrt{x}+12}=\frac{\sqrt{x}+14}{\sqrt{x}+38}$. $x=1$.

12. $\frac{7-\sqrt{x}}{11-\sqrt{x}}=\frac{12-\sqrt{x}}{20-\sqrt{x}}$. $x=4$.

13. $\frac{\sqrt{x+a}}{\sqrt{x-b}}=\frac{\sqrt{x-a}}{\sqrt{x+b}} \cdot \sqrt{\frac{bm}{a^2-x}} + \sqrt{\frac{ed}{a^2+x}} = \sqrt[4]{\frac{4bdem}{a^4-x^2}} \quad (14.)$

$$14. \frac{x-a}{9\sqrt{a}+5\sqrt{x}} = \frac{\sqrt{x}+\sqrt{a}}{5}. \quad x=a.$$

$$15. \frac{3\sqrt[3]{x}}{4} + 0.125\sqrt{x} + 23 = 30 - 7\frac{\sqrt{x}}{0.16} + 0.875\sqrt{x}. \quad x=.0256.$$

16. $\sqrt[6]{-a+b^3+\sqrt[3]{(x+ab)(a-b)+b^3}} = \sqrt[3]{b}$; raising each member to the sixth power, we obtain

$$-a+b^3+\sqrt[3]{(x+ab)(a-b)+b^3}=b^3;$$

or, dropping the b^3 from each member, and transposing a to the second member, $\sqrt[3]{(x+ab)(a-b)+b^3}=a$; whence, cubing each member,

$$(x+ab)(a-b)+b^3=a^3; \text{ or, } (x+ab)(a-b)=a^3-b^3;$$

dividing each member by $a-b$, $x+ab=a^3+ab+b^3 \therefore x=a^3+b^3$.

$$17. \sqrt[3]{\frac{a^6-b^6}{(a-b)x}+b^3}=a. \quad \text{Ans. } x=\frac{a^3+b^3}{a-b}.$$

$$18. \sqrt[3]{\frac{a^6-b^6}{(a-b)x}-b^3}=a. \quad x=a^3+ab+b^3.$$

$$19. \sqrt[3]{7-\frac{8a-b}{x-c}}=2. \quad x=c+b-8a.$$

$$20. \sqrt{3+\sqrt[4]{\frac{1-c^2}{x}}}+c+1295=3. \quad x=1+c.$$

SOLUTION OF PROBLEMS.

359. An algebraic problem requires us to find one or more quantities conforming to certain requirements. These requirements are called the *Conditions* of the problem. *Explicit* conditions are such as are stated formally and distinctly in the problem; *Implicit* conditions are such as are not distinctly stated, but left to be inferred from others in which they are *implied*.

The solution of a problem consists of:

1°. The statement of the problem.

2°. The solution of the resulting equation or equations.

1°. **The Statement** of the problem is the expression, in algebraic language, of the conditions of the problem, *i. e.*, the expression of the relations of the known quantities to the required quantities. The statement is always one equation or several equations. When, as in the problems upon which we are about to enter, but one unknown quantity occurs, the statement consists of one equation. Translating the conditions of a problem, from the common language of the problem into algebraic language, is a process for which no precise rule can be given. Accurate knowledge of the subject-matter and skillful application of this knowledge must be the student's main reliance, but the following directions will be found of material assistance:

DIRECTIONS.

I. *Get a thorough understanding of the meaning of the problem.*

Ascertain the relations between the known and the required quantities, i. e., determine what steps would be taken in verifying the value of the unknown quantity (or quantities), if this value were already known.

II. *Denote one of the unknown quantities by one of the closing letters of the alphabet, and find, from the conditions of the problem, expressions for the other unknown quantities, if there be any others.*

If fractional parts of any unknown quantity are to be taken, it is best to denote this quantity by a letter multiplied by the lowest common denominator of the fractional units. We thus, to a great extent, avoid fractions in the equation.

III. *Express, in algebraic language, the relations of the given to the required quantities.*

2°. **The Solution of the equation or equations.**

If, as in the following problems, but one equation forms the statement, this equation may be solved by (357).

The treatment of problems giving rise to several equations will be taken up later.

PROBLEMS.

1. Divide the number 56 into two parts, such that twice the first part shall exceed the second part by 7.

Let x = the first part.

Then $56-x$ = the second part.

Twice the first part is $2x$; and since this exceeds the second part by 7, if 7 be added to the second part, the result will equal $2x$.

Hence $2x = 56 - x + 7$; transposing,

$3x = 56 + 7$; and reducing,

$3x = 63 \therefore x = 21$.

Since 21 is the first part, $56-21=35$ is the second.

To verify this result:

Twice 21 is 42, and 42 exceeds 35 by 7, as required by the problem.

2. There are two numbers of which the sum is 110; the first number, increased by 4, equals three times the second diminished by 2. What are the numbers?

Let x = the first number. Then $110-x$ = the second number.

$x+4$ = the first number increased by 4,

$3(110-x-2)$ = three times the second diminished by 2.

Hence, from the conditions,

$x+4 = 3(110-x-2) = 3(108-x)$, whence

$x+4 = 324-3x$

$4x = 320$

$x = 80$ = the first number,

$110-x = 30$ = the second.

To verify this result: $80+4=84$, and $30-2=28$, and 3 times 28=84.

3. The difference between two numbers is 20; if, from three times the less, 12 be subtracted, the remainder will be greater by 15 than twice the second number. Required the numbers.

Ans. 67 and 87.

4. Find two numbers, such that the sum shall be 100, and three times the greater, diminished by 5 times the less shall also equal 100.

Ans. 25 and 75.

5. Find two numbers such that the difference shall be 14, and three times the greater shall exceed, by 3, six times the less.

Ans. 13 and 27.

6. Divide 177 into two parts, such that one half of the greater, divided by 5, may be less by one than one seventh of the smaller.

7. Divide the quantity c into two parts, such that a times the first shall exceed b times the second by d .

Let x = the first part. Then $c-x$ = the second part.

And from the conditions

$$\begin{aligned} ax &= b(c-x) + d; \text{ whence} \\ ax &= bc - bx + d \\ ax + bx &= bc + d \\ (a+b)x &= bc + d \\ x &= \frac{bc+d}{a+b} \end{aligned}$$

This problem is a *generalization* of the preceding, and the result, the value of x , is, as it were, a formula from which, by proper substitutions, we may obtain the answer to any problem of this kind. Thus, if $c=56$, $a=2$, $b=1$, and $d=7$, we have problem 1, and $x = \frac{bc+d}{a+b}$ becomes $\frac{1 \times 56 + 7}{2 + 1} = \frac{63}{3} = 21$. In a similar way we may apply the formula to any other numerical problem.

8. Divide 234 into two parts, such that the difference between the parts shall be 10.

9. The sum of two numbers is 128, and the difference 36. Required the numbers. *Ans.* 82 and 46.

10. The sum of two numbers is s , and the difference d . Required the numbers.

Let x = the less number. Then $x+d$ = the greater, and from the conditions

$$\begin{aligned} x + x + d &= s; \quad 2x + d = s, \text{ or } 2x = s - d; \\ \text{whence } x &= \frac{s}{2} - \frac{d}{2}; \quad x + d = \frac{s}{2} - \frac{d}{2} + d = \frac{s}{2} + \frac{d}{2}. \end{aligned}$$

This is a *general form* of problem, of which the preceding is a special example, the results being obtainable by the substitution of 128 for s and 36 for d . Similarly we may substitute for s and d any numbers whatever, and the answer to the problem may be obtained by simply

making the substitutions in the values of x and $x+d$. Since s and d represent any quantities whatever, we see that: *Given the sum and the difference of any two numbers, the GREATER equals half the sum PLUS half the difference, and the LESS equals half the sum MINUS half the difference.*

11. Find the numbers of which the sum is 175 and the difference 35. Find the numbers of which the sum is 934 and the difference 118.

12. A drover bought 5468 head of stock. He bought twice as many horses as cattle, lacking 1, and three times as many sheep as horses. How many of each did he buy?
Ans. 1215 horses, 608 cattle, 3645 sheep.

13. On a farm are horses, cattle and sheep. The cattle and sheep together number 258. The number of sheep exceeds the number of cattle by 84, and the number of horses is one third the number of cattle. Required the number of each. *Ans.* 29 horses, 87 cattle, and 171 sheep.

14. A hare, pursued by a hound, has a start of 75 of its own leaps. The hare makes 10 leaps while the hound makes 7, but 5 leaps of the hare are equivalent to only 2 leaps of the hound. How many leaps must the hound make to overtake the hare?

Let x = number of leaps made by the hound. Since the hare makes 10 leaps while the hound makes but 7, the hare makes $\frac{10}{7}$ leaps while the hound makes but 1. Hence it will make $\frac{10}{7} \times x$ leaps while the hound makes $1 \times x$, or simply x , leaps. $75 + \frac{10x}{7}$ is, then, the number of leaps of the hare in going over the whole distance to be traversed by the hound. Since the leaps of the hound are not equal to the leaps of the hare, we cannot place $x = 75 + \frac{10x}{7}$, for the first member would represent a number of units, and the second a number of units of a different kind. We know, however, that 5 leaps of the hare equal 2 of the hound, and hence that each leap of the hound equals $\frac{5}{2}$ of a leap of the hare. Hence x leaps of the hound equal $\frac{5x}{2}$ leaps of the hare. Hence

$\frac{5x}{2}$ is the distance traversed by the hound, if the leap of the hare be taken as the unit. Hence

$$75 + \frac{10x}{7} = \left\{ \begin{array}{l} \text{whole distance traversed by the hare, the leap of the hare} \\ \text{being the unit, and} \end{array} \right.$$

$$\frac{5x}{2} = \left\{ \begin{array}{l} \text{whole distance to be traversed by the hound, expressed in} \\ \text{terms of the same unit.} \end{array} \right.$$

$$\begin{aligned} \text{Hence, } \frac{5x}{2} &= 75 + \frac{10x}{7} \\ 35x &= 1050 + 20x \\ 15x &= 1050 \\ x &= 70. \end{aligned}$$

Or let $7x$ = number of leaps of the hound.

Then $10x$ = number of leaps of the hare in the same time.

Then $7x:10x+75::2:5$; $35x=20x+150$; $15x=150$; $x=10 \therefore 7x=70$, the number of leaps of the hound.

15. Divide the number 750 into three parts, such that the second may be $1\frac{1}{2}$ times as great as the first, and the third $2\frac{1}{2}$ times the first.

Let $2x$ represent the first part. Then

$3x$ represents the second part, and

$5x$ represents the third part, whence

$$2x+3x+5x=750$$

$$10x=750$$

$x=75$, and 150, 225, and 375 are the required parts.

16. Divide 340 into three parts, such that the first shall be to the second as 2 is to 5, and the second to the third as 5 is to 13.

Ans. 34, 85 and 221.

17. \$1760 is to be divided among three partners, so that the second may receive $1\frac{1}{2}$ times as much as the first, and the third $1\frac{1}{4}$ times as much as the second. What are the shares?

Ans. \$320, \$560 and \$880.

18. The denominator of a certain fraction exceeds the numerator by 10. If 6 be added to each term of this fraction the value of the fraction will be $\frac{49}{119}$. What is the

fraction?

Ans. $\frac{1}{11}$.

19. In a certain regiment four officers were assigned to each company, when it was found that the number of officers was too small by 6, to allow this apportionment. Three officers were then assigned to each company, when it was found that 5 officers remained for extra duty. How many companies were in the regiment? *Ans.* 11.

20. A party of boys having received some dimes, wished to divide them so as to give each boy the same sum. If they had 12 dimes more, they could have received seven apiece, and, on giving each boy 6, there were three dimes left. Required the number of boys.

21. A speculator has, in warehouse, grain as follows: $\frac{3}{5}$ of the whole, plus 100 bu., is wheat; $\frac{1}{2}$ of the remainder, plus 200 bu., is corn; of what remains, $\frac{1}{2}$ is rye, and there are 1200 bu. of barley. How much grain in the warehouse? How much of each kind?

Ans. 2750 bu. of wheat.

22. $\frac{1}{10}$ of a mixture is hydrochloric acid, $\frac{1}{5}$ plus 20 gr. sulphuric acid, $\frac{1}{2}$ of the remainder, minus 16 gr., nitric acid, and of what remains one half is orange syrup and 180 gr. distilled water. Required the amount of each ingredient and the total.

Total 540 gr.

23. A courier rode from Mons toward Vichy, at the rate of m miles per hour. Half way between the towns he met with an accident, so that, although he continued his journey without delay, he was able to ride only c miles per hour for the remainder of the journey. The whole time occupied in traveling from Mons to Vichy was a hours. Required the distance between the towns.

Ans. $\frac{2amc}{m+c}$

24. Of the money expended by the Red Cross Mining Co. during 1874, one fifth, less \$5000, was paid for tunneling and timbering, the tunneling costing 60% more than

the timbering. Of the remainder, one half increased by \$10,000 was paid for new machinery, 35%, less \$2500, for wages, and the other \$12,000 were expended in acquiring certain mineral lands. Required the different items of expenditure.

25. A privateer having disposed of prizes, the money obtained was divided as follows: $\frac{3}{5}$ of the whole, plus \$30,000, was paid as lawyers' fees; of the remainder, $\frac{1}{10}$ and \$7000 in addition, was paid for repairing the ship; of the remaining sum, $\frac{1}{2}$, with \$5000 additional, was divided among the officers, and the residuary \$50,000 was distributed to the crew. Required the whole amount, and the separate items of disbursement.

Total \$400,000.

26. A detachment of troops leaves X—— at 3 A. M. to attack W——, 123 miles distant, and proceeds at the rate of $2\frac{1}{2}$ miles per hour (allowing for stoppages). At 4 P. M. of the same day the pickets before W—— learn of the intended attack, and at 5 P. M. a detachment sets out to meet the enemy. If this detachment travel at the rate of 3 miles per hour, at what time will the two forces meet?

Ans. 9 A. M. next day.

27. The troops from X—— being surprised, flee, panic-stricken, toward X——. Rallying his troops at a village near X——, the commanding officer finds that $\frac{3}{5}$ of his command, plus 2500 men, is missing, and $\frac{1}{5}$ of those present, minus 1000, are wounded; of the remainder, $\frac{1}{2}$ are cavalry and 12,000 infantry. How many troops were in the detachment originally?

Ans. 30,000.

28. A river flows $\frac{1}{10}$ of its length, minus 10 miles, through a barren, mountainous country; half the remaining distance, plus 15 miles, through broad pasture lands; of the remainder of the course, $\frac{2}{3}$ lies through a low, but populous country, and the remaining 75 miles is through a marsh. Required, the length of the river and the divisions of its course.

Ans. Total length, 300 miles.

29. A merchant has 100 pieces of cloth, of four kinds. The number of pieces of the second is greater by 18 than $\frac{1}{2}$ the number of pieces of the first kind; the number of pieces of the third is less by 10 than twice the number of the second; and there are four times as many of the fourth kind as of the first. Required, the number of pieces of each kind.

Ans. 10 of the 1°, 20 of the 2°, 30 of the 3°, 40 of the 4°.

30. At what time between 6 and 7 o'clock will the hour and minute-hands of a watch be together?

Ans. $32\frac{8}{11}$ minutes past 6.

31. A contractor hired a certain number of men to do a piece of work which he was bound to finish in thirty working days. He agreed to pay them \$4 per day, but at the end of the first week, the men struck, and demanded \$5 per day, to which demand the contractor was obliged to accede. Had he paid the men \$4 per day, as agreed, he would have gained \$2,000 on the contract, but in fact, he lost \$1,000. Required, the number of men employed. *Ans.* 125.

32. A vintner has two casks: one contains a gallons of wine and b gallons of water; the other contains a' gallons of wine and b' gallons of water. He wishes to draw from them wine and water to form a mixture containing c gallons of wine, and c gallons of water, (c being greater than at least two of the quantities, a , b , a' , b'). How much will he draw from each cask?

From the first $\frac{(a+b)(a'+b')-2a'(a+b)}{(a'+b')a-(a+b)a'} c$ gals.

33. A grocer has two kinds of sugar; the one worth a cents per pound, and the other worth b cents. He wishes to form a mixture of m pounds, such that any pound of it may be sold, without either gain or loss, for c cents. Required, the number of pounds of each kind of sugar that he should take.

Ans. $\frac{a-c}{a-b}m$; $\frac{c-b}{a-b}m$.

34. A can do a piece of work in a days, B can do the same work in b days, and C the same work in c days. How many days will they require in which to complete this work if all three work together?

$$\text{Ans. } \frac{abc}{ab+bc+ac}.$$

35. Two chariots are driven in the same direction on a race-track a miles in circumference, the one at the rate of n miles per hour, the other pursuing at the rate of m miles per hour. When will they be together for the first time, second time, etc., if they are c miles apart when starting?

$$\text{Ans. In } \frac{c}{m-n}, \frac{c+a}{m-n}, \frac{c+2a}{m-n}, \dots \text{ hours.}$$

36. Divide the number n into three parts, such that the second shall be $\frac{b}{a}$ times as large as the first, and the third $\frac{c}{b}$ times as large as the second.

$$\text{Ans. } \frac{a}{a+b+c}n, \frac{b}{a+b+c}n, \frac{c}{a+b+c}n.$$

37. Divide the number N into four parts, such that the first increased by a , the second diminished by a , the third multiplied by a , and the fourth divided by a , shall be equal.

$$\text{Ans. } \frac{aN}{(a+1)^3} - a; \frac{aN}{(a+1)^3} + a; \frac{N}{(a+1)^3}; \frac{a^3N}{(a+1)^3}.$$

38. A merchant with a certain capital expends annually $\$nc$, and each year augments the unexpended part of his capital by $\frac{1}{n}$ th of itself. At the end of the third year his original capital is increased p -fold. Required, the original capital.

39. The circumference of each of the fore wheels of a carriage is f feet, and of each of the hind wheels h feet. How far must the carriage be drawn in order that each of the fore wheels shall make a revolutions more than each of the hind wheels?

$$\text{Ans. } \frac{afh}{3(h-f)} \text{ yds.}$$

II. EQUATIONS WITH MORE THAN ONE UNKNOWN QUANTITY.

360. If we have but a single equation involving two unknown quantities, we cannot obtain the value of either of them. For if we attempt to solve the equation with respect to either unknown quantity, we shall obtain in one member this quantity, and in the other an expression involving the second, and hence unknown. Thus, in the equation, $4x+2y=7$ (1), $2y=7-4x$ (2), and hence $y=\frac{7-4x}{2}$ (3).

Now, if we know the value of x , y becomes known, but not otherwise. It is plain, however, that no matter what the value of x , y is always equal to $\frac{7-4x}{2}$, and if we substitute

for y , $\frac{7-4x}{2}$ in (1), the two members will reduce to the same quantity, no matter what the value of x . If, however, we assign to x a definite numerical value, $\frac{7-4x}{2}$ will reduce to some number, and this number, taken with the assigned value of x , will satisfy the given equation. Thus, if we make x equal to 1, 2, 3, 4, etc., in succession,

we have y equal to $\frac{3}{2}$, $-\frac{1}{2}$, $-\frac{5}{2}$, $-\frac{9}{2}$, etc., and (1) becomes.....

4 +	$\frac{6}{2}$	=7, or	4 + 3 = 7
8 -	$\frac{2}{2}$	=7, or	8 - 1 = 7
12 -	$\frac{10}{2}$	=7, or	12 - 5 = 7
16 -	$\frac{18}{2}$	=7, or	16 - 9 = 7
etc., etc.,			

and any value of y , taken with the corresponding value of x , will satisfy the given equation. Any value of x , taken with the value of y obtained by substituting the value of x in (3), forms a *Set* of values. As we can assign to x an infinite number of values, every one of which, substituted in (3), gives a corresponding value of y , which, with the assigned value of x will satisfy (1), there are an infinite number of sets of values of x and y that will satisfy

the given equation. Such an equation, i. e., one satisfied by an infinite number of sets of values of the unknown quantities, is said to be *Indeterminate*. The values of the unknown quantities cannot be *fixed* or *determined*. Thus we cannot say that $x=7$ or 8, for $x=26$ or 32, just as well. We cannot say that $y=2$, or 3, or 7, or 10, or 12, for it equals 200, or 75, or 36, or 512, or anything else, just as well.

We might have solved the given equation with reference to x , thus, $x = \frac{7-2y}{4}$ (4), and, by substituting any number whatever for y , deduce a value of x , which taken with this value of y would satisfy the given equation. As we may treat any equation containing two unknown quantities in a similar way, we see that

Any single equation containing two unknown quantities is indeterminate.

Suppose we have the equation $7x+3y+2z=12$ (5), whence $x = \frac{12-3y-2z}{7}$ (6). If, in (6), we substitute, in succession,

1, 2, 3, 4, 5 etc., for y and z , we obtain values, $1, \frac{2}{7}, -\frac{3}{7}$, etc.,

which, substituted for x , in (5), must, with the values as-

signed to y and z , satisfy the equation; thus:
$$\left\{ \begin{array}{l} 7+3+2=12 \\ 2+6+4=12 \\ -3+9+6=12 \\ \text{etc., etc.} \end{array} \right.$$

Similarly, if we have any equation containing more than one unknown quantity, we may deduce an expression for one of the unknown quantities, in terms of the others. By assigning values at pleasure to these others we may deduce the corresponding value of the first, and thus obtain a *Set* of values, that will satisfy the given equation, and, as we may assign any values whatever to these others, and therefrom deduce the corresponding value of the first, there will be an infinite number of sets of values that will satisfy the given equation. Hence the given equation will be indeterminate; i. e., in general,

361. *Any single equation, containing more than one unknown quantity, is indeterminate.*

362. Let us now resume the equation $4x+2y=7$ (1), whence $y=\frac{7-4x}{2}$ (3). If the value of x is fixed in any way, the value of y will be fixed, for, on substituting the value of x in (3), y must be equal to $\frac{7-4x}{2}$, which will then be a *fixed number*. Again, if y could have an infinite number of values, as a value of x could be found corresponding to each, x would have an infinite number of values, which is contrary to the original supposition. If, then, the value of either x or y be fixed or limited, the value of the other will also be fixed.

Now, suppose that we have a new equation, $4y-x=5$ (7), and suppose that the values of x and y in (7) are the same as the values of x and y in (1). From (7) we have $y=\frac{5+x}{4}$ (8), whence, the values of y in (7) and (1) being the same, $\frac{5+x}{4}=\frac{7-4x}{2}$ (9), or $5+x=14-8x$, or $9x=9$, or $x=1$. In order then, that the values of y in (7) and (1) be the same, x must be 1, under which supposition,

$$y=\frac{7-4x}{2}=\frac{5+x}{4}=\frac{3}{2}.$$

The values, $y=\frac{3}{2}$ and $x=1$, will, then, satisfy both (7) and (1); and these equations are not otherwise satisfied. These equations are said to be simultaneous, and in general,

363. Simultaneous Equations *are equations in which the values of the unknown quantities are the same in all, at the same time.*

We see, from what precedes, that, in order to obtain fixed values for two unknown quantities, we must have *two* simultaneous equations involving these quantities.

In a similar manner we may show that for the determination of *three* unknown quantities *three* equations are

needed; for determining *four* unknown quantities *four* independent equations, and, in general, for determining n unknown quantities, n independent equations are needed.

ELIMINATION.

364. Elimination is the process of combining several equations, involving several unknown quantities, in such a manner as to deduce therefrom a less number of equations, involving a less number of unknown quantities.

(A.) TWO SIMULTANEOUS EQUATIONS ARE GIVEN.

365. I.—To deduce a Rule for elimination by Addition or Subtraction.

Let $2x+3y=7$ (1)
 $3x-6y=14$ (2) be the given equations. Examining these equations, we see that if the coefficients of x were the same in (1) and (2), and we should subtract (1) from (2), the resulting equation would involve y alone; i. e., x would be eliminated. Let us, then, transform the equations so that the coefficients of x may be equal in the two. Multiplying both members of (1) by 3, and both numbers of (2) by 2, we have $6x+9y=21$ (3)
 $6x-12y=28$ (4), and subtracting, member from member, (3) from (4), we obtain $-21y=7$, which equation involves only y , x having been eliminated by subtraction.

Again, it may be seen that if the coefficients of y in the two equations (1) and (2) had been the same, we might, by *adding* member to member, (1) and (2), have obtained an equation not involving y . We may, therefore, eliminate y by transforming the given equations so as to make the coefficients of y equal, and adding the transformed equations member to member. Multiplying both members of (1) by 2, we have $4x+6y=14$ (5). Adding (5) and (2), member to member, we have, $7x=28$, an equation involving only x , y having been eliminated by Addition.

If the given equations contain more than two unknown quantities, any one of these quantities may be eliminated

by Addition or Subtraction in a manner exactly similar to that here employed.

From the foregoing considerations we deduce the

RULE.—*Transform the two given equations so that the coefficients of the quantity to be eliminated may be numerically equal in the two. If these coefficients have unlike signs, add the equations, member to member; if they have like signs, subtract either equation from the other, member from member.*

366. We can always effect the required transformation by finding the lowest common multiple of the coefficients of the quantity to be eliminated, and multiplying the members of each equation by the quotient of the lowest common multiple divided by the coefficient of this unknown quantity in the other. In particular cases, however, simpler methods of transforming will often suggest themselves, but for these no rule can be given. It is well for the student to accustom himself to perform the operations for elimination mentally, writing only the results.

This method of elimination is often called the *English Method*.

EXAMPLES.

Deduce a single equation from each of the following groups:

1. $5x - 3y = 26$ *Ans.* $-44y = 18$
 $3x + 7y = 12$ or, $44x = 218$
2. $13y - 2x = 17$
 $5y + 3x = 9$
3. $\frac{2x}{6} - \frac{3y}{8} + 5 = 20.$
 $\frac{3x}{4} - \frac{9y}{4} = 7.$ *Ans.* $5x = 332.$
4. $\frac{27x}{5} - 36y = 12.$
 $\frac{15y}{4} - 11x = 7.$
5. $\frac{3y - x}{5} + 3\frac{17 - y}{2} = 12; \frac{15x}{7} - \frac{3y}{5} = 2x - 7.$

367. II.—To Deduce a Rule for elimination by Substitution.

From either of two simultaneous equations we may, as in (360), find an expression for one of the unknown quantities in terms of the others, and (363) we may substitute this for the unknown quantity, in the other, thus obtaining an equation free from this unknown quantity. If, for example, we have $\begin{matrix} 4y-x=5 & (1) \\ 2y+4x=7 & (2) \end{matrix}$, from (2) we obtain $x = \frac{7-2y}{4}$, and as the value of x must be the same in (1) and (2), we may substitute this expression for x in (1), thus obtaining $4y - \frac{7-2y}{4} = 5$, an equation involving only y , x having been eliminated by Substitution. Hence, for elimination by Substitution we have the

RULE.—Find from one of the given equations an expression for the quantity to be eliminated, in terms of the other quantities. Substitute this expression for the unknown quantity itself in the other equation.

EXAMPLES.

- | | |
|--|----------------------------|
| 1. $\begin{matrix} 13x - y = 62 \\ 5x + 11y = 77 \end{matrix}$ | <i>Ans.</i> $138x = 605$. |
| 2. $\begin{matrix} 4x + 2y = 34 \\ x + 3y = 21 \end{matrix}$ | <i>Ans.</i> $10y = 50$. |
| 3. $\begin{matrix} 12y - 7x = 16 \\ 2y + x = 20 \end{matrix}$ | <i>Ans.</i> $26y = 156$. |
| 4. $\begin{matrix} 3y + 2x = 31 \\ 5y + 4x = 55 \end{matrix}$ | <i>Ans.</i> $y = 7$. |
| 5. $\begin{matrix} 7y + 3x = 66 \\ 11y - 6x = 93 \end{matrix}$ | <i>Ans.</i> $25y = 225$. |

368. III.—To Deduce a Rule for elimination by Comparison.

From each of two simultaneous equations we may, as in (360), obtain an expression for one of the unknown quantities to be eliminated, in terms of the others. From the definition (363) of simultaneous equations, we may equate

(i. e., connect by the sign of equality) these two expressions. As neither contains the unknown quantity for which the expressions have been found, the resulting equation will be free from this quantity. Thus if $\frac{7x+3y=22}{-5x+2y=10}$ (1) (2), we have from (1), $y=\frac{22-7x}{3}$, and from (2), $y=\frac{10+5x}{2}$, and since the values of y in (1) and (2) are the same, $\frac{22-7x}{3}=\frac{10+5x}{2}$, which equation contains only x , y having been eliminated by Comparison.

Hence for elimination by Comparison, we have the

RULE.—Find from each equation an expression for the quantity to be eliminated, in terms of the others. Equate these expressions.

EXAMPLES.

$$1. \quad \begin{array}{l} 15x+4y=19 \\ 16y-9x=13. \end{array} \quad \text{Ans. } 69x=63.$$

$$2. \quad \begin{array}{l} \frac{11x}{7}+6=2y-3 \\ 3x+7y=91. \end{array} \quad \text{Ans. } 17x=119.$$

$$3. \quad \begin{array}{l} 13y-7z=12 \\ 17y-3z=28. \end{array} \quad \text{Ans. } 80z=160.$$

$$4. \quad \begin{array}{l} 9x-2y=10 \\ 3x-7y=12. \end{array} \quad \text{Ans. } 19y=-26.$$

$$5. \quad \left. \begin{array}{l} \frac{x}{2}+\frac{y}{3}=1 \quad x=2\left(1-\frac{y}{3}\right) \\ \frac{x}{3}+\frac{y}{4}=1 \quad x=3\left(1-\frac{y}{4}\right) \end{array} \right\} \quad \begin{array}{l} 2-\frac{2y}{3}=3-\frac{3y}{4}; \text{ or} \\ -8y=12-9y; \end{array}$$

369. IV.—French Method. To Deduce a Rule for elimination by the use of a Multiplier of which the value is to be determined.

Let the given equations be $\begin{cases} 4y-3x=11 & (1) \\ 2y+x=13 & (2) \end{cases}$. If, now, we multiply both members of (2) by any quantity, as m , we have $2my+mx=13m$ (3). Adding (1) and (3), member to

member, we have $(4+2m)y+(m-3)x=11+13m$ (4). If, now, the coefficient of x in (4) were 0, the term involving x would reduce to 0, *i. e.*, x would not occur in the equation. This coefficient, $m-3$, will reduce to 0 if $m=3$. Hence, since m may be any quantity whatever, we place it equal to 3, and (4) becomes, $10y=50$, an equation containing only y , *i. e.*, x has been eliminated. If we had wished to eliminate y , we should wish the term involving y to disappear from (4), which requires, then, $m=-2$, and under this supposition (4) becomes $-5x=-15$, or $5x=15$.

We could have obtained like results by subtracting (3) from (1), member from member, for we would thus have obtained $(4-2m)y-(3+m)x=11-13m$ (5), in which the substitution of $m=2$ gives $-5x=-15$, or $5x=15$, and the substitution of $m=-3$ gives $10y=50$.

Since we can assign to m , arbitrarily, any value whatever, we may call it an *arbitrary multiplier*. From the foregoing considerations we deduce the following

RULE.—*Multiply both members of one of the given equations by an arbitrary multiplier. Add this transformed equation to the other given equation, member to member, or subtract it from this other, member from member. Place (in the result) the coefficient of the quantity to be eliminated equal to 0, and substitute the resulting value of the arbitrary multiplier in the equation obtained by addition or subtraction.*

This process is often called *Elimination by the use of Undetermined multipliers*.

EXAMPLES.

- | | | |
|-------------------------------|----------------------------|----------------------------------|
| 1. $2x+7y=16$
$3y-4x=2$ | 2. $6x+3y=21$
$4x-5y=7$ | <i>Ans. $y=1$.</i> |
| 3. $24x-25y=17$
$13x+2y=5$ | 4. $7y+2x=29$
$3y-x=10$ | <i>Ans. $13y=49$.</i> |

(B.) WHEN MORE THAN TWO SIMULTANEOUS
EQUATIONS ARE GIVEN.

370. To Deduce a Rule for elimination when more than two equations are given.

$$\text{Let } 3x-4y+z = 4 \quad (1)$$

$$2x+3y-3z=32 \quad (2)$$

$$-x+7y-z=40 \quad (3) \text{ and let it be required to deduce a}$$

single equation, containing but one unknown quantity.

A little reflection shows that we cannot combine (1), (2), and (3), at random, for if we should combine (1) and (3), eliminating z , and then (1) and (2), eliminating y , the resulting equations would still have three unknown quantities, for z would occur in the equation arising from combining (1) and (2) so as to eliminate y .

If, however, we combine (1) and (2), eliminating z , the resulting equation contains only x and y ; and if we now combine (1) and (3), eliminating z , the resulting equation contains only x and y . We thus have two equations containing only x and y , which we may combine so as to eliminate either, and thus perform the operation required.

$$\text{Again, take the equations } \begin{cases} x+y+z+w=8 & (1) \\ 3x-2y-z+5w=12 & (2) \\ 4x+3y+2z-w=16 & (3) \\ 2x-y+5z+3w=20 & (4) \end{cases}$$

We cannot combine at random, for the reason already indicated, but if we combine (1) with (2), (3), and (4), in succession, eliminating w each time, we have one equation for each combination, *i. e.*, three in all, and these three will contain only x , y , and z . The first of these we may then combine with each of the others, eliminating z each time, and obtaining one equation for each combination, *i. e.*, two altogether, containing only x and y , combining which, we get a single equation involving one unknown quantity.

In general, if we have n simultaneous equations, containing m unknown quantities, we cannot combine them at random, for a reason already indicated. If, however, we

combine the first with each of the $n-1$ others, eliminating each time the same unknown quantity, we shall obtain as many equations as there are combinations, i. e., $n-1$, and as we have each time eliminated the same unknown quantity, there will be but $m-1$ unknown quantities left. We have now a group of $n-1$ equations, with $m-1$ unknown quantities, from which, in the manner just indicated, we may deduce $n-2$ equations containing $m-2$ unknown quantities, and by continuing the operation we may finally obtain a single equation. Hence the

RULE.—Combine one of the given equations with each of the others, eliminating the same unknown quantity each time. Treat the resulting group of equations in the same manner, and continue the operation until a single equation is obtained, or until the operation has been carried as far as is desired.

371. Scholium.—If $m=n$, the final equation will evidently have but one unknown quantity. If $m>n$, it will have more than one, and if $m<n$ it will have none.

372. Scholium II.—If each equation does not contain all the unknown quantities, the work of elimination may often be simplified. For example: let $\begin{cases} x+y=18 & (1) \\ y+z=9 & (2) \\ z+x=15 & (3) \end{cases}$ be the given equations. From (1) and (2) we obtain $x-z=9$, and combining this with (3) we obtain $2x=24$. Labor-saving devices will often thus suggest themselves to the quick or practiced eye.

373. The method of elimination by the use of an arbitrary multiplier may easily be extended to the case of three or more equations. Thus, take the equations

$$2x+3y+z=11 \quad (1)$$

$$x-2y+2z=3 \quad (2)$$

$$-11x-6y+3z=-14 \quad (3)$$

Multiplying both members of (1) by m , and of (2) by n , the equality of the members will in neither case be destroyed, and we shall have

$$2mx+3my+mz=11m \quad (4)$$

$$nx-2ny+2nz=3n \quad (5)$$

Adding (4), (5), and (3), member to member, the results must be equal in the two members, and we have

$$(2m+n-11)x+(3m-2n-6)y+(m+2n+3)z=11m+3n-14 \quad (6)$$

If, now, such values be given to m and n as to make the coefficients of x and y each equal to zero, these quantities will be eliminated from the equation.

$$\text{In this case, } \begin{array}{l} 2m+n-11=0 \\ 3m-2n-6=0 \end{array} \quad \text{or, } \begin{array}{l} 4m+2n-22=0 \quad (7) \\ 3m-2n-6=0 \quad (8) \end{array}$$

$$\text{Adding (7) and (8), member to member, } \begin{array}{r} 7m \quad -28=0 \\ \text{or, } m \quad -4=0 \end{array}$$

$$\text{whence, } m \text{ must be equal to 4. If } m=4, \begin{array}{l} 2m+n-11=0 \\ \text{or, } 8+n-11=0 \\ \text{or, } n-3=0 \\ \text{or, } n=3 \end{array}$$

Substituting the values of m and n in (6) we find

$$(4+6+3)z=44+9-14, \quad 13z=53-14=39;$$

an equation involving only z .

In a precisely similar manner this method may be extended to any number of equations.

EXAMPLES.

$$1. \quad \begin{cases} 2y-3x+4z=82 \\ 5y-2x+7z=67 \\ 2y-11x+15z=2. \end{cases}$$

$$2. \quad \begin{cases} 6x+7y-10z=2 \\ 4x-y+z=21 \\ 3x+2y-3z=6. \end{cases}$$

$$3. \quad \begin{cases} 3x-2y+z=13 \\ 5x+2y-3z=17 \\ 2x-3y+2z=9. \end{cases}$$

$$\text{Ans. } 3x=18.$$

$$4. \quad \begin{cases} \frac{2x}{5} + \frac{3y}{2} - \frac{z}{4} = 7 \\ x - \frac{y}{3} + z = 8 \\ \frac{3x}{10} + \frac{2y}{6} + \frac{3z}{4} = 11 \end{cases}$$

$$5. \quad \begin{cases} 3w-4z+x+3y=0 & (1) \\ 4w+2y-3x-6z=4 & (2) \\ w-2x+2z-y=4 & (3) \\ 2w+8z-3y+3x=3 & (4) \end{cases}$$

$$\text{From (1) and (3), we have, } \begin{array}{l} 2x=8z-6w-6y \\ 2x=w+2z-y-4 \end{array} \quad \begin{array}{l} (5) \\ (6) \end{array}$$

$$\text{From (2) and (4), we have, } \begin{array}{l} 3x=4w+2y-6z-4 \\ 3x=3-2w-8z+3y \end{array} \quad \begin{array}{l} (7) \\ (8) \end{array}$$

From (5) and (6), we have, $\frac{8z-6w-6y=w+2z-y-4}{-6z+7w+5y=+4}$, or (9)

From (7) and (8), we have $\frac{4w+2y-6z-4=3-2w-8z+3y}{6w+2z-y=7}$, or (10)

In (5), $x=4z-3w-3y$ and in (7) $x=\frac{4w+2y-6z-4}{3}$. Hence

$$4z-3w-3y=\frac{4w+2y-6z-4}{3}, \text{ whence}$$

$$13w-18z+11y=4 \quad (11)$$

From (9), (10), (11), we find $\begin{cases} 2z=\frac{7w+5y-4}{3} \end{cases}$ (12)

$$2z=\frac{13w+11y-4}{9} \quad (13)$$

$$2z=7+y-6w \quad (14)$$

From (12) and (13), we find, $\frac{7w+5y-4}{3}=\frac{13w+11y-4}{9}$,

whence $\begin{cases} 8w+4y=8 \\ 2w+y=2 \end{cases}$ (15)

From (12) and (14) we find $\frac{7w+5y-4}{3}=7+y-6w$, whence

$$25w+2y=25 \quad (16)$$

From (15), $y=2-2w$ (17). From (16), $y=\frac{25-25w}{2}$ (18)

From (17) and (18) we find, $2-2w=\frac{25-25w}{2}$, whence $21w=21$

$$6. \begin{cases} 3w+2x-6z-3y=3 & (1) \\ 2w+10z+2y-3x=8 & (2) \\ x-2w-12z-4y=-1 & (3) \\ 4w+4z+x=3 & (4) \end{cases} \quad \begin{array}{l} \text{From (2) we find} \\ y=\frac{8+3x-2w-10z}{2}; \\ \text{substituting in (1) we} \end{array}$$

obtain, $3w+2x-6z-\frac{24+9x-6w-30z}{2}=3$; whence,

$$12w-5x+18z=30 \quad (5)$$

Substituting the value of y in (3) we have:

$$\begin{aligned} x-2w-12z-16-6x+4w+20z &= -1, \text{ or} \\ 2w-5x+8z &= 15 \end{aligned} \quad (6)$$

From (6): $2w=5x-8z+15$, which, substituted in (5), gives $30x-48z+90-5x+18z=30$; or,

$$\begin{aligned} 25x-30z &= -60 \\ 5x-6z &= -12 \end{aligned} \quad (7)$$

Substituting in (4) the value of $2w$, we have

$$\begin{aligned} 10x-16z+30+x+4z &= 3; \text{ or,} \\ 11x-12z &= -27 \end{aligned} \quad (8)$$

From (7) we have $6z=12+5x$, which, substituted in (8) gives $11x-24-10x=-27$; or, $x=-3$.

$$7. \begin{cases} 2w-3x+2y-z=1 & (1) \\ 3w+5z-3y=4 & (2) \\ w+3z+y+2x=4 & (3) \\ 4x-3w-y-2z=1 & (4) \end{cases}$$

Multiplying both members of (1) by 2, $4w-6x+4y-2z=2$

Multiplying both members of (3) by 3, $3w+6x+3y+9z=12$

$$\begin{array}{r} \text{Adding, member to member,} \\ \hline 7w \qquad +7y+7z=14 \\ w \qquad +y+z=2 \end{array} \quad (5)$$

From (3) we have, $2w+4x+6z+2y=8$

From (4), $-3w+4x-2z-y=1$

$$\text{Whence,} \quad \begin{array}{r} 5w \qquad +8z+3y=7 \end{array} \quad (6)$$

Adding (2) and (6), member to member, we have,

$$8w+13z=11 \quad (7)$$

Multiplying both members of (5) by 3, and subtracting, member from member, from (6), we have:

$$\begin{array}{r} 5w+8z+3y=7 \\ 3w+3z+3y=6 \\ \hline 2w+5z=1 \end{array} \quad (8)$$

Multiplying both members of (8) by 4, we have:

$$8w+20z=4.$$

Subtracting this equation from (7), member from member, we have:

$$-7z=+7.$$

$$8. \begin{cases} 2x-5y+3z=1 & (1) \\ -3x+7y+z=19 & (2) \\ 8x-17y-20z=-115 & (3) \end{cases}$$

From (1) and (2) we obtain, by multiplying by arbitrary multipliers, m and n ,

$$2mx-5my+3mz=m \quad (4)$$

$$-3nx+7ny+nz=19n \quad (5)$$

Adding (4), (5) and (3), member to member,

$$\begin{aligned} (2m-3n+8)x-(5m-7n+17)y+(3m+n-20)z= \\ m+19n-115 \end{aligned} \quad (a)$$

Placing the coefficients of x and y each equal to zero, (so as to eliminate x and y), we have

$$\begin{array}{rcl} 2m-3n+8=0 & (6) & \\ 5m-7n+17=0 & (7) & \end{array} \quad \text{or,} \quad \begin{array}{rcl} 10m-15n+40=0 & & \\ 10m-14n+34=0 & & \end{array}$$

$$\text{whence,} \quad \frac{-n+6=0}{-n+6=0}$$

and $n=6$. If $n=6$, $3n=18$, and $2m-18+8=0$; or, $2m-10=0$; whence, $2m=10$ and $m=5$. Substituting the values of m and n in (a), we have

$$(15+6-20)z=5+114-115, \text{ or,} \\ z=4.$$

$$9. \quad \begin{cases} 3x+2y-4z-3w=10 \\ 4x+5y-6z-7w=17 \\ 2x+y+z+5w=30 \\ x+y-7z+11w=12 \end{cases}$$

$$10. \quad \begin{cases} x-ay+a^2z=a^3 \\ x-by+b^2z=b^3 \\ x-cy+c^2z=c^3 \end{cases}$$

$$11. \quad \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{3}{4}; & \frac{1}{y} + \frac{1}{z} = \frac{5}{12}; & \frac{1}{z} + \frac{1}{w} = \frac{7}{24}; \\ \frac{1}{w} + \frac{1}{v} = \frac{9}{40}; & \frac{1}{v} + \frac{1}{x} = \frac{3}{5}. \end{cases}$$

SOLUTION OF SIMULTANEOUS EQUATIONS.

374. The process of *Elimination* generally precedes the *Solution* of simultaneous equations, but the two operations are totally distinct. It is often desirable to eliminate when we are wanting in either the desire or the ability to effect the solution.

375. To Deduce a Rule for finding the values of the unknown quantities from several simultaneous equations.

Since we know how to solve a single equation containing but one unknown quantity, if, from the given equations, we can deduce a single one containing but one unknown quantity, we can find the value of that unknown quantity. If we substitute this value in an equation containing only one other unknown quantity, we may, as already shown, find the value of this second quantity. Substituting in an equation containing only one other unknown quantity, we

may find the value of this third, and so on, until the values of all the quantities have been found. Thus, taking the

$$\text{equations: } \begin{cases} 3x-4y+z=4 & (1) \\ 2x+3y-2z=32 & (2) \\ -x+7y-z=40 & (3) \end{cases} \text{ we deduce:}$$

$$8x-5y=40 \quad (4) \\ 2x+3y=44 \quad (5) \text{ and finally } 17y=136, \text{ whence } y=8.$$

Substituting this value of y in (5), we obtain $2x+24=44$, or $2x=44-24=20$, whence $x=10$, and substituting these values of x and y in (1) we obtain $30-32+z=4$; or, $-2+z=4$; or, $z=4+2$; or, $z=6$; and the values $x=10$, $y=8$, $z=6$, are the values required.

In a similar manner we might treat the case of any group of simultaneous equations having the number of unknown quantities equal to the number of equations, as, in this case, we may always deduce a single equation, containing but one unknown quantity. Hence the

RULE.—*Apply to the given equations the rule for elimination, continuing the process until a single equation containing but one unknown quantity is obtained. Solve this equation, and, by substituting this root in one of the immediately preceding equations, containing but two unknown quantities, find the value of a second unknown quantity. Substitute the values now found in one of the equations containing but three unknown quantities, and thus find the value of a third quantity. Continue the process until the values of all the unknown quantities have been found.*

376. If the given equations contain more unknown quantities than there are equations, the result of the elimination is (371) an equation containing more than one unknown quantity, and therefore indeterminate.

If, on the other hand, we have a less number of unknown quantities than of equations, the process of elimination enables us to get rid of all the unknown quantities, and the resulting equations, containing only known quantities, are called *Equations of Condition*. They express the relations that must obtain among the known quantities in order that

the given equations may be satisfied; in other words, they express the *conditions* under which alone the given equations can be satisfied. But if equations are satisfied only on condition that the known quantities entering some have certain relations to those entering others, the equations are not wholly independent; some depend on the others. Hence such equations are called *Dependent* equations. In general:—

Independent Equations are equations expressing essentially different conditions, so that no one can be derived from any of the others.

Dependent Equations are such as do not all express essentially different and unrelated conditions, and they can not all be satisfied unless the known quantities have certain relations to each other. Dependent equations are always so related that one or more may be derived from the others.

Equations of Condition are such as contain only known quantities, and express relations of those quantities essential to the solution of some dependent equations.

ILLUSTRATION. Let $\begin{cases} ax+by=c & (1) \\ ax-by=d & (2) \\ bx=2ay & (3) \end{cases}$ be the given equations.

From (1) and (2) we deduce $x = \frac{c+d}{2a}$; $y = \frac{c-d}{2b}$, whence, substituting in (3), $\frac{b(c+d)}{2a} = \frac{a(c-d)}{b}$ (4), an *equation of condition*, expressing the relations that must subsist among the known quantities in order that all the given equations may be satisfied. If values are to be found that will satisfy all, (3) may evidently be derived from (1) and (2).

For example, let $a=6$, $b=3$, whence (4) becomes $\frac{c+d}{8} = c-d$. Now, if $c=9$, and $d=7$, (4) is satisfied, and $x = \frac{4}{3}$, $y = \frac{1}{3}$, and (1), (2), and (3),

$$\text{become } 6\left(\frac{4}{3}\right) + 3\left(\frac{1}{3}\right) = 9 \text{ or } 8 + 1 = 9$$

$$6\left(\frac{4}{3}\right) - 3\left(\frac{1}{3}\right) = 7 \text{ or } 8 - 1 = 7$$

$$3\left(\frac{4}{3}\right) = 12\left(\frac{1}{3}\right) \text{ or } 4 = 4.$$

But if $c=15$ and $d=9$, (4) is not satisfied, the values $x=\frac{24}{12}=2$, and $y=\frac{6}{6}=1$, satisfy (1) and (2), $\left\{ \begin{matrix} 12+3=15 \\ 12-3=9 \end{matrix} \right\}$ but do not satisfy (3), which becomes $6=12$.

EXAMPLES.

Find the values of the unknown quantities in the following equations:

$$(1.) \begin{cases} 3x+2y=31 & x=5 \\ 5x-y=17 & y=8 \end{cases} \quad (2.) \begin{cases} 22x-13y=-15 & x=7. \\ 11x-3y=38 & y=13. \end{cases}$$

$$(3.) \begin{cases} 2x - \frac{7-y}{2} = 4y + \frac{x-y}{6} & x=11. \\ \frac{x+y}{2} - \frac{x-y}{3} = \frac{5y-1}{4} & y=5. \end{cases}$$

$$(4.) \begin{cases} \frac{x-y}{2} + \frac{3(x-y)}{2} + 10 = \frac{5y-6}{3} + 11 & x=5. \\ 8y+7x-24 = \frac{16y-13}{2} + \frac{10x-15}{2} & y=3. \end{cases}$$

$$(5.) \begin{cases} \frac{2x-10}{2} - \frac{5y+17}{11} + 14 = \frac{11y-32}{10} + y + 7 & x=27. \\ \frac{23-2y}{4} + \frac{4x}{27} + 5y-8 = \frac{2x+6}{2} + 2y + \frac{7}{4} & y=12. \end{cases}$$

$$(6.) \begin{cases} 2x+3y+4z=56 & x=5. \\ 5x-4y+6z=43 & y=6. \\ 3x-7y+5z=8 & z=7. \end{cases} \quad (7.) \begin{cases} 12x-8y+z=35 & x=8. \\ 5x-6y+7z=63 & y=9. \\ 2x+y+3z=58 & z=11. \end{cases}$$

$$(8.) \begin{cases} \frac{2x-3}{3} + \frac{y-2}{2} - z = -\frac{5z-14}{9} & x=6. \\ \frac{x+y}{2} + \frac{y+z}{3} + 7 = \frac{z-5}{5} + \frac{z+8}{2} + z & y=8. \\ \frac{x}{2} + \frac{y+1}{3} + \frac{z+2}{4} = 9 & z=10. \end{cases}$$

$$(9.) \begin{cases} \frac{x}{2} + \frac{y}{2} = 10 & x=8. \\ \frac{y}{3} + \frac{z}{3} = 12 & y=12. \\ \frac{x}{4} + \frac{z}{4} = 8 & z=24. \end{cases}$$

(10.)

$$\begin{cases} \frac{2}{x} - \frac{2}{y} = \frac{1}{4} \\ \frac{2}{y} + \frac{3}{z} = \frac{1}{2} \\ \frac{3}{z} - \frac{2}{x} = \frac{3}{4} \end{cases}$$

(12.)

$$\begin{cases} \frac{x}{a} + \frac{y}{a} = 1 & x = -a \\ \frac{y}{a} + \frac{z}{a} = 2 & y = 2a \\ \frac{z}{a} + \frac{w}{a} = 3 & w = 3a \\ \frac{w}{a} - \frac{x}{a} = 4 & z = 0 \end{cases}$$

(11.)

$$\begin{cases} 2x - 3y + 5z + w = 48 & x = 20. \\ 2x - 3z + 2y - w = 35 & y = 1. \\ x - y + 2w - 10z = 1 & z = 2. \\ 4y - x + w + 11z = 7 & w = 1. \end{cases}$$

(13.)

$$\begin{cases} x + 2y + z = 8 & x = 1. \\ 2y - x + 2z = 9 & y = 2. \\ 5x + y + z + w = 14 & z = 3. \\ x + y + 7z + 5w = 44 & w = 4. \end{cases}$$

(14.)

$$\begin{aligned} 3x + 2y + z &= 27 \quad (1) & \text{From (2) and (4), we have } -6x + 4y + z &= -10 \quad (6) \\ 12x - 7y + w &= 10 \quad (2) & \text{" (3) " (5),} & 2x - y + 4z &= 33 \quad (7) \\ 3y - 2z + 2v &= 21 \quad (3) & \text{" (1) " (6),} & 8y + 3z &= 64 \quad (8) \\ 6x - 3y + z + w &= 20 \quad (4) & \text{" (6) " (7),} & y + 13z &= 109 \quad (9) \\ x + y + z + v &= 27 \quad (5) & \text{" (8) " (9),} & 101z &= 808 \end{aligned}$$

whence $z = 8$

Substituting in (9), $y + 104 = 109$. $\therefore y = 5$. Substituting in (7), $2x - 5 + 32 = 33$. $\therefore x = 3$; whence, (5), $3 + 5 + 8 + v = 27$. $\therefore v = 11$. Finally, from (2), $36 - 35 + w = 10$. $\therefore w = 9$.

$$(15.) \quad \begin{cases} x + y = 5 & x = 1 \\ y + z = 11 & y = 4 \\ z + w = 17 & z = 7 \\ w + v = 23 & w = 10 \\ x + v = 14 & v = 13 \end{cases}$$

$$(16.) \quad \begin{cases} x + y + z = 21 & t = 4. \\ y + z + v = 26 & v = 11. \\ z + v + w = 28 & w = 9. \\ v + w + t = 24 & x = 6. \\ t + z + y = 19 & y = 7. \\ t + v + w = 24 & z = 8. \end{cases}$$

PROBLEMS.

1. Find two numbers, such that five times the first, diminished by four times the second, shall be equal to 2, and four times the second, diminished by three times the first, shall be equal to 18.

Let x = the first number, and y = the second number.

$$\text{Then, from the conditions, } \begin{cases} 5x - 4y = 2 & (1) \\ 4y - 3x = 18 & (2) \end{cases}$$

Eliminating y , $2x = 20$. $\therefore x = 10$. Substituting in (1), $50 - 4y = 2$; $4y = 48$. $\therefore y = 12$. The required numbers are 10 and 12.

2. Arthurst and Brown, working together, can do a certain amount of copying in 12 hours; Brown and Thomas can do the same amount in 10 hours, while Arthurst and Thomas require 15 hours to complete the work. How many hours would be required by each to do this work?

Let x = the number of hours required by Arthurst;

y = the number of hours required by Brown;

z = the number of hours required by Thomas.

Then $\frac{1}{x}$ denotes the part performed in 1 hour by Arthurst;

$\frac{1}{y}$ denotes the part performed in 1 hour by Brown;

$\frac{1}{z}$ denotes the part performed in 1 hour by Thomas.

Since Arthurst and Brown, working together, perform the work in 12 hours, in one hour they perform $\frac{1}{12}$ of it. Similarly, Brown and Thomas perform $\frac{1}{10}$ in one hour, and Arthurst and Thomas $\frac{1}{15}$ in one hour.

Hence $\frac{1}{x} + \frac{1}{y} = \frac{1}{12}$ (1) Eliminating y from (1) and (2), we have

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{12} \quad (1) \quad \frac{1}{z} - \frac{1}{x} = \frac{1}{60} \quad (4); \text{ eliminating } x \text{ from (3)}$$

$$\frac{1}{y} + \frac{1}{z} = \frac{1}{10} \quad (2) \quad \text{and (4), } \frac{2}{z} = \frac{1}{12} \therefore z = 24.$$

$$\frac{1}{x} + \frac{1}{z} = \frac{1}{15} \quad (3)$$

Substituting in (4), we find $\frac{1}{x} = \frac{1}{40} \therefore x = 40$, and finally, substituting in (1), $y = 17\frac{1}{2}$.

Hence Arthurst alone would do the work in 40 hours, Brown in $17\frac{1}{2}$ hours, and Thomas in 24 hours.

3. Find two numbers, such that three times their sum, diminished by twice the less, shall be equal to 17, and four times their difference, increased by three times the greater, shall be equal to 27. Ans. 5 and 2.

4. Find a fraction such that, if 3 be subtracted from the numerator, the value of the fraction will be $\frac{2}{3}$; and if 4 be subtracted from the denominator the value will be $\frac{1}{2}$.

5. There are two grammar schools in the —th ward. Double the number of pupils in the first exceeds four times the number in the second, by 600; and $\frac{2}{3}$ of the number in

the first exceeds $\frac{4}{5}$ of the number in the second, by 100. Required, the number of pupils in each.

Ans. 1500 and 600 respectively.

6. Find two numbers, such that a times the first, increased by b times the second, shall be equal to c , and d times the first, increased by e times the second, shall be equal to f .

7. A farmer has two bins full of corn. If from the first he take twice the contents of the second, there will remain in the first $\frac{1}{2}$ of the difference in the original contents of the two bins minus 10 bushels. If the second bin held $2\frac{3}{4}$ bushels less than it now holds, seven times its contents would be precisely equal to three times the contents of the first when full. Required, the capacity of each bin.

Ans. 75 bu. and 35 bu. respectively.

8. A number is expressed by two digits. If 18 be added to the number, the sum will be expressed by the same digits reversed; if 54 be subtracted from the number, the remainder will be greater by 6 than the number of units in the given number. What is the number? *Ans.* 68.

9. A grocer has two kinds of sugar. If he mixes 2 lbs. of the first with 3 lbs. of the second, the mixture will be worth 11 cents $\frac{1}{2}$ lb; if he mixes 5 lbs. of the first with 7 lbs. of the second, the mixture will be worth $11\frac{1}{4}$ cents $\frac{1}{2}$ lb. Required the value of a pound of each kind.

Ans. $12\frac{1}{2}$ cents and 10 cents respectively.

10. A gardener has a certain number of roses and of pinks. If there were five times as many roses and three times as many pinks, the number of the former would exceed the number of the latter by 50. If, on the other hand, there were three fourths as many roses and one third as many pinks, the former would exceed the latter by 25. Required the number of each.

Ans. 100 roses, 150 pinks.

11. There are two numbers, each expressed by two digits, and in each the number of units is one half the number of tens. One third the sum of the numbers is greater by 1 than five times the number of tens in the first, and the difference between the numbers exceeds three-sevenths of the greater number, by 3. Required the numbers.

Ans. 42 and 21.

12. The army of Cyrus at the battle of Cunaxa consisted of Persians and Greek auxiliaries, in all 130,000, minus the number of Greeks. Had the Greeks been three times, and the Persians four times, as numerous as they in fact were, the army would have numbered 355,000, plus six times the number of Greeks. Required the number of Greeks and of Persians. *Ans.* Greeks, 15,000, Persians, 100,000.

13. The population of M—— is 12 times the excess of the number of whites in the town over the number of colored persons. Two fifteenths of the number of whites is less, by 7000, than one fifth of the number of colored persons. Required the number of whites, and the population of the town. *Ans.* Whites, 195,000; population, 360,000.

14. A man has two houses and furniture worth \$7000. When the furniture is in the first house, the value of this house is less by \$3000 than double the value of the second. When, on the other hand, the furniture is in the second house, three times the value of this house is less by \$14,000 than four times the value of the first house. Required the value of each house. *Ans.* 1°, \$20,000; 2°, \$15,000.

15. A fruiterer had two baskets containing apples. Having sold one third the number in the second basket he found that there were left in the two baskets 390 apples. He then sold one half of those remaining in the second, and one fifth the number in the first, and found that the number then remaining in the two baskets was 270. How many did each basket contain at first? *Ans.* 1°, 250; 2°, 210.

16. The steamship *Cyprus* left Queenstown for Newfoundland on her first trip, and then returned to Queenstown twice. On her second arrival in Queenstown, it was found that all the trips eastward were made in equal times, and the like was true of the westward trips, and the whole time actually occupied in sailing was 30 days. During the westward trips she met strong head winds, and made only $46\frac{2}{3}$ miles, while on her eastward trips she made, on an average, $53\frac{1}{4}$ miles. Assuming the sailing distance at 3000 miles per trip, what was the time occupied in making each trip eastward and each trip westward?

Ans. 7 and 8 days respectively.

17. A library is furnished exclusively with English and French books. If, from one third the whole number of books, one half the number of French books be subtracted, the remainder will be 1000. Again, if to three times the number of French works 1500 be added, the sum will be three halves of the number of books in the library. Required, the number of French and of English works.

Ans. 4000 French; 5000 English.

18. A gentleman, visiting his farm, found that if to one half the number of acres of swamp land, two thirds of the number of acres of meadow land were added, the sum would be 240 acres; also, that one half the meadow land exceeded four fifths of the swamp land by 86 acres. Required, the number of acres of meadow and of swamp.

Ans. 300 meadow; 80 swamp.

19. Find three numbers, such that the sum shall be 30; half the sum of the first two, increased by five times the third, shall equal 51; and the first, increased by five times the difference between the second and third, shall equal 22.

Ans. 12; 10; 8.

20. Find three numbers, such that the sum of the quotients arising from dividing the first by a and the second by b , the sum of the quotients of the second by b and the third

by c , and the sum of the quotients of the first by a and the third by c , shall each be equal to 4. *Ans.* $2a$; $2b$; $2c$.

21. A laborer is engaged for d days on condition that he is to receive $\$a$ for every day that he works, and is to forfeit $\$b$ for every day that he is idle. At the end of the d days he receives $\$c$. Required, the number of days on which he worked, and the number on which he was idle.

$$\text{Ans. } \frac{c+db}{a+b}; \frac{da-c}{a+b}.$$

22. A merchant has three vessels containing wine. If he adds a gallons to the first, it will contain as much as both of the others. If he adds a gallons to the second, it will then contain twice as much as the first and third (together) originally contained. Finally, if he adds a gallons to the third, it will contain three times as much as the first and second together originally contained. Required, the number of gallons in each vessel.

$$\text{Ans. } \frac{a}{11}; \frac{5a}{11}; \frac{7a}{11}.$$

23. A sold his farm for a certain sum; B sold his for the same sum. A received in payment a note at a months, which amounted, when due, to $\$b$. B received a note for the same sum, with interest at the same rate, but in c months the amount of the note was $\$d$. For how much was each farm sold? What was the common rate of interest?

$$\text{Ans. } \frac{cb-ad}{c-a}; \frac{1200(d-b)}{cb-ad}.$$

24. An artisan wishes to make an ornament containing $10\frac{3}{4}$ oz. of gold, 7 oz. of silver, and $4\frac{1}{2}$ oz. of copper. He is to take his material from three ingots, each composed of these metals melted together. In the first ingot there are 10 oz. of gold, 6 oz. of silver, and 4 oz. of copper. In the second there are 12 oz. of gold, 9 oz. of silver, and 3 oz. of copper. The third contains 7 oz. of gold, 4 oz. of silver, and 5 oz. of copper. How many ounces must he take from the first,

second, and third ingots respectively, to form the ornament desired?

Let x = the number of ounces taken from the first;

y = the number of ounces taken from the second;

z = the number of ounces taken from the third.

Since in the 20 oz. of the first ingot there are 10 oz. of gold, in 1 oz. there must be $\frac{10}{20}$ oz. of gold, or $\frac{1}{2}$ oz. of gold. Hence in x ounces of the first ingot there must be $\frac{x}{2}$ oz. of gold. Similarly, in x oz. of this ingot there must be $\frac{6x}{20}$ oz., or $\frac{3x}{10}$ oz., of silver, and $\frac{4x}{20}$ oz., or $\frac{x}{5}$ oz., of copper.

Reasoning in a manner precisely similar, we find that in y oz. of the second ingot, there must be $\frac{12y}{24}$ oz., or $\frac{y}{2}$ oz., of gold, $\frac{3y}{8}$ oz. of silver, and $\frac{y}{8}$ oz. of copper. And in the z oz. from the third, there are $\frac{7z}{16}$ oz. of gold, $\frac{4z}{16}$, or $\frac{z}{4}$ oz. of silver, and $\frac{5z}{16}$ oz. of copper.

The desired ornament, then, as it contains x oz. from the first ingot, y oz. from the second, and z oz. from the third, must contain $\frac{x}{2} + \frac{y}{2} + \frac{7z}{16}$ ounces of gold. By the conditions, it must contain $10\frac{1}{2}$ ounces of gold.

Hence $\frac{x}{2} + \frac{y}{2} + \frac{7z}{16} = \frac{43}{4}$ or $8x + 8y + 7z = 172$ (1).

Reasoning in a similar manner, we obtain for the number of ounces of silver in the ornament, $\frac{3x}{10} + \frac{3y}{8} + \frac{z}{4}$ which, from the conditions, must equal 7, or $\frac{3x}{10} + \frac{3y}{8} + \frac{z}{4} = 7$, or $12x + 15y + 10z = 280$ (2).

Lastly, we have, for the number of ounces of copper in the ornament,

$\frac{x}{5} + \frac{y}{8} + \frac{5z}{16}$ which must be equal to $4\frac{1}{2}$; or, $\frac{x}{5} + \frac{y}{8} + \frac{5z}{16} = \frac{17}{4}$; or, $16x + 10y + 25z = 340$ (3).

Multiplying both members of (1) by 3, and both members of (2) by 2, and subtracting the former from the latter, member from member, we have, $6y - z = 44$ (4). Multiplying both members of (1) by 2, and subtracting (3), member by member, from the resulting equation, we obtain $6y - 11z = 4$ (5). Subtracting (5) from (4), member from member, we have, $10z = 40$. $\therefore z = 4$. Substituting this value of z in (4), we obtain $6y = 48$. $\therefore y = 8$. Substituting in (1), we obtain $8x + 64 + 28 = 172$; $8x = 80$. $\therefore x = 10$. Hence the jeweller must take 10 oz. from the first ingot, 8 oz. from the second, and 4 oz. from the third.

25. A number of men, women, and children, took passage in the steamer for L—; if half the number of women and children were added to the whole number of men on board, the sum would be 425; if one fifth the number of women and children were subtracted from three halves of the number of men, the remainder would be 210; finally, if to three times the number of men, twice the number of women were added, the sum would exceed twice the number of children by 100. Required the number of men, of women, and of children.

Ans. 200 men; 100 women; 350 children.

26. A boy bought apples at 2 cents, pears at 2 cents, and oranges at 5 cents, apiece, paying 51 cents for all. Had he paid 3 cents apiece for the apples, 5 cents for the pears, and 8 cents for the oranges, his fruit would have cost him 90 cents. Another boy bought apples at 4 cents, pears at 5 cents, and oranges at 9 cents apiece, buying as many of each as the first boy, and paying \$1.05 for the lot. Required the number of apples, of pears, and of oranges.

Ans. 12 apples, 6 pears, 3 oranges.

27. In a library are works on history, poetry, and philosophy; $\frac{1}{3}$ of the number of works on history, increased by $\frac{1}{2}$ of the number of works on poetry, exceeds by 100 the number of philosophical works. Twice the number of the first kind, diminished by four times the number of the third kind, is less by 50 than three times the number of the second kind. Lastly, the number of works of the first two kinds is five times the number of works of the third kind. Required the number of works of each kind in the library.

Ans. 750, 250, and 200 respectively.

28. A prize of \$70,000 was divided among the officers and crew of a privateer, and the lawyers employed to distribute the money: $\frac{1}{7}$ of the lawyers' share, increased by $\frac{2}{3}$ of the seamen's share, was less by \$5000 than the officers' share; twice the lawyers' share, diminished by three times

the officers' share, was equal to $\frac{2}{3}$ of the seamen's share, and 4 times the seamen's share, increased by $\frac{1}{2}$ of the officers' share, was equal to $\frac{7}{2}$ of the lawyer's share, increased by the seamen's share. Required the amount of each share.

Ans. Lawyers', \$35,000; officers', \$20,000; crew, \$15,000.

29. Three men have a piece of work to do. A and B together can do it in m days; B and C together can do it in n days, and A and C together in b days. In how many days can each do the work? In how many days can they finish it if all work together?

Let x = the number of days in which A alone can do it;

y = " " B " "
 z = " " C " "

Since A and B can do the whole work in m days, in one day they can together do the part $\frac{1}{m}$; and as A alone can do the whole work in x

days, in one day he performs the part represented by $\frac{1}{x}$. Similarly B in one day performs the part represented by $\frac{1}{y}$. The sum of the parts that they can perform singly obviously equals the part that they can perform, working together. Hence, $\frac{1}{x} + \frac{1}{y} = \frac{1}{m}$ (1), and by a precisely similar

course of reasoning we obtain $\frac{1}{y} + \frac{1}{z} = \frac{1}{n}$ (2), and $\frac{1}{x} + \frac{1}{z} = \frac{1}{b}$ (3). Subtracting (2) from (1), member from member, we obtain $\frac{1}{x} - \frac{1}{z} = \frac{1}{m} - \frac{1}{n}$ (4).

Adding (3) and (4), member to member, we obtain $\frac{2}{x} = \frac{1}{m} - \frac{1}{n} + \frac{1}{b}$,

whence $x = \frac{2mn}{mn + nb - mb}$. Subtracting (4) from (3), member from member, we obtain $\frac{2}{z} = \frac{1}{b} + \frac{1}{m} - \frac{1}{n}$; whence, $z = \frac{2mn}{mn + mb - nb}$; and in a similar

manner we obtain $y = \frac{2mn}{mb + nb - mn}$.

Now let t = the number of days in which they can finish it if all work together. If we call the work 1, or unity, A does $\frac{1}{x}$, B $\frac{1}{y}$, and C $\frac{1}{z}$, in 1 day. Hence all, working together, do $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ in 1 day. But $\frac{1}{x} + \frac{1}{y} = \frac{1}{m}$. Hence, all working together perform $\frac{1}{m} + \frac{1}{z}$ in 1 day, and therefore in t days they perform $t \left(\frac{1}{m} + \frac{1}{z} \right)$; and since in t days they

finish the work, t times the part done in 1 day must equal the whole work, or unity, i. e., 1. Hence, $t\left(\frac{1}{m} + \frac{1}{z}\right) = 1$, or, substituting the value of z , as just found,

$$t\left(\frac{1}{m} + \frac{mn+mb-nb}{2mnb}\right) = 1; \text{ or, } t\left(\frac{2mnb+m^2n+m^2b-mnb}{2mnb}\right) = 1; \text{ or,}$$

$$t\left(\frac{nb+mn+mb}{2nb}\right) = 1. \therefore t = \frac{2nb}{nb+mn+mb}$$

30. A farmer finds, on counting his horses, cows, goats and sheep, that if the number of horses be increased by the number of cows and three times the number of goats, the sum will be less by 10 than four times the number of sheep; if five times the number of horses be diminished by twice the number of cows and goats together, the remainder will be less by 90 than the number of sheep; if three times the excess of the number of goats over the number of cows be added to seven times the number of horses, the sum will exceed, by 100, three times the number of sheep; finally, if 10 be subtracted from twice the number of goats and sheep together, the remainder will equal seven times the number of horses, diminished by one half of the number of goats. Required the number of horses, of cows, of goats and of sheep.

Ans. 40; 50; 60; 70.

31. In a wheel are three iron spokes. The longer distance, on the edge of the wheel, from the end of the first spoke to the end of the third is 20 c. m.; the longer distance, on the edge, from the second to the first, is 21 c. m.; and the longer distance, on the edge, from the third to the second, is 19 c. m. Required the shorter distance, on the edge of the wheel, from the first to the second, from the second to the third, and from the third to the first.

Ans. 9, 11 and 10 c. m., respectively.

32. A boatman who rows a miles up a stream, and back, in b hours, finds that he can row c miles with the current in the time that he would require to row d miles against it.

Required, the time spent in rowing up the stream, and that spent in rowing down; also the rate of the current.

$$\text{Ans. } \frac{bc}{c+d}; \frac{bd}{c+d}; \frac{a}{2bcd}(c^2-d^2).$$

83. A railroad train left Fulton for Troyes, and after traveling for a hours, met with an accident that delayed it b hours, after which it went on at $\frac{m}{p}$ ths of its original rate, and arrived at Troyes c hours behind time. Had the accident, and consequent delay, happened d miles nearer Troyes, the train would have arrived at this town e hours sooner than it actually did. Required, the distance from Fulton to Troyes, and the rate of speed of the train before the accident.

Let x represent the number of miles between the towns, and y the number of miles traveled by the train in 1 hour, before the accident; whence $\frac{my}{p}$ is the number of miles traveled per hour, after the accident.

Then $\frac{x}{y}$ is the number of hours regularly allowed for the trip; $x-ay$ is the number of miles traveled after the accident, and $\frac{px-apy}{my}$ is the number of hours consumed in traveling after the accident.

$$\text{Hence, } \frac{px-apy}{my} + a + b = \frac{x}{y} + c. \quad (1)$$

Again, $x-(ay+d)$ would be the number of miles traveled after the accident, if the delay had occurred d miles nearer Troyes. $a+\frac{d}{y}$ would be the number of hours consumed in traveling at the original rate, and $\frac{px-apy-pd}{my}$ the number occupied in traveling at the reduced rate.

$$\text{Hence, } \frac{px-apy-pd}{my} + a + \frac{d}{y} + b = \frac{x}{y} + c - e. \quad (2)$$

Clearing (1) and (2) of fractions, and eliminating x , we find

$$y = \frac{p-m}{em}d; \text{ whence, } x = \frac{ap-m(a+b-c)}{em}d.$$

34. A freight train from A—, after traveling 2 hours, met with an accident that delayed it 3 hours, and then, proceeding at $\frac{4}{5}$ of its original rate, arrived at its destination, B—, 5 hours behind time. Had the accident and conse-

quent detention, occurred 48 miles nearer B——, the train would have been but 4 hours and 24 minutes late. Required, the distance from A——to B——, and the rates of speed before and after the accident. *Ans.* 200; 20; 16.

35. A and B start together to ascend a mountain. A would have reached the summit 70 minutes before B, but, taking a by-path, misses his way, and goes $2\frac{1}{2}$ miles and back needlessly, walking at $\frac{2}{3}$ of his former rate. Regaining the main road, he resumes his former pace, and reaches the summit 10 minutes before B. C starts 45 minutes later than A and B, and, walking at a uniform rate of $2\frac{1}{2}$ miles per hour, reaches the summit 15 minutes after B. At what rates did A and B walk, and what was the distance from the point of starting to the summit of the mountain.

Ans. 3, 2, and 7 miles, respectively.

SECTION II.

EQUATIONS OF THE SECOND DEGREE.

377. An Equation of the Second Degree is one in which the greatest sum of the exponents of the unknown quantities in any term is 2.

An equation with one unknown quantity is of the second degree, when the highest exponent of the unknown quantity in any term is 2. We proceed to the discussion of this class of equations.

378. To Deduce a Rule for reducing any equation of the second degree to the form $x^2+2px=q$.

Let $\frac{a}{b}x^2-mx-n=fx^2+\frac{k}{g}x+\frac{g}{3}$ be any equation of the second degree, with but one unknown quantity.

Clearing this equation of fractions, we have

$$3agx^2-3mbgx-3bng=3bfgx^2+3bkx+bg^2.$$

Transposing, so that all the terms containing x may be in the first member, and all the others in the second,

$$3agx^2-3bfgx^2-3bmgx-3bkx=3bng+bg^2, \text{ or} \\ 3(ag-bfg)x^2-3b(mg+k)x=gb(3n+g).$$

Dividing both members of this equation by $3g(a-bf)$, the equality not being destroyed, we shall have

$$x^2-\frac{b(mg+k)}{g(a-bf)}x=\frac{b(3n+g)}{3(a-bf)}.$$

If, now, we place $-\frac{b(mg+k)}{g(a-bf)}=2p$ and $\frac{b(3n+g)}{3(a-bf)}=q$, we have, $x^2+2px=q$.

Since any equation of the second degree may be treated similarly, we deduce the following

RULE.—*Clear the equation of fractions. Collect in the first member all terms containing the unknown quantity, and in the second all the other terms, and reduce each member to its simplest form.*

Collect into one term all terms involving the square of the unknown quantity; do the same with those involving the first power. Divide by the coefficient of the square of the unknown quantity.

379. Scholium.—The form $x^2+2px=q$ is sometimes changed to $x^2+2px-q=0$, which is evidently equivalent to it.

380. The coefficient $2p$ may, in general, be positive, negative, or zero; so also the term q . If $2p=0$, the equation takes the form $x^2=q$, and is called an **Incomplete Equation**, as it wants a term; and if $2p$ is not equal to 0, the equation is called a **Complete Equation** of the second degree.

SOLUTION OF EQUATIONS.

381. As the solution of *Incomplete Equations* is simpler than the solution of *Complete Equations*, we shall take up *Incomplete Equations* first.

To Deduce a Rule for solving an Incomplete Equation of the Second Degree.

From (378), we know that the given equation may be reduced to the form $x^2+0.x=q$; or $x^2=q$. Since the square roots of equals are equal, $\sqrt{x^2}=\sqrt{q}$, i. e., $x=\sqrt{q}$, or since the square root always has the sign $\pm$, $x=\pm\sqrt{q}$. The equation, then, has *two roots*, viz. $+\sqrt{q}$ and $-\sqrt{q}$, numerically equal, having contrary signs.

Hence, to solve any incomplete equation, we have the

RULE.—Reduce the equation to the form $x^2=q$. The first root of the equation will be the square root of the second member, the second will be minus the square root of the second member.

Corollary I.—The sum of the two roots equals zero.

Corollary II.—The product of the two roots equals the second member with its sign changed.

If the second member is a perfect square, we may obtain the roots exactly, whether they be numerical or algebraic. If the second member is an algebraic expression, and an imperfect square, we may reduce the radical to its simplest form by the rules already given. If the second member is numerical, and an imperfect square, we may approximate, as nearly as we please, to the true roots of the equation, by any of the rules (already demonstrated) for finding approximate roots.

382. It might be supposed that, as $\sqrt{x^2}$ has two signs, + and —, we have four roots, but this is not the case; we gain nothing by taking into account the double sign of x , for, considering $\sqrt{x^2}$ as + or —, we have

$$\left. \begin{array}{l} +x = +\sqrt{q} \text{ (1)} \\ +x = -\sqrt{q} \text{ (2)} \\ -x = +\sqrt{q} \text{ (3)} \\ -x = -\sqrt{q} \text{ (4)} \end{array} \right\} \begin{array}{l} \text{In these expressions (1) and (4) are evidently equivalent, since either may be derived from the other by} \\ \text{changing the signs of both members of that other;} \\ \text{similarly with (2) and (3); nothing, therefore, being} \\ \text{gained by considering the double sign of } \sqrt{x^2}, \text{ this is never done; we} \\ \text{write simply } x = +\sqrt{q} \text{ and } x = -\sqrt{q}. \end{array}$$

EXAMPLES.

Solve the following equations:

$$1. \quad \frac{3}{8}x^2 + 15 = \frac{3x^2 - 6}{2}$$

Clearing of fractions, we have, $3x^2 + 120 = 12x^2 - 24$;
 Transposing, $3x^2 - 12x^2 = -120 - 24$;
 Uniting terms, $-9x^2 = -144$;
 Reducing, $x^2 = 16$.

Hence $x = +4$ and $x = -4$; +4 and —4 are, then, the required roots.

To verify:
$$\frac{3}{8}(4)^2 + 15 = \frac{48}{8} + 15 = 6 + 15 = 21$$

$$\frac{3(4)^2 - 6}{2} = \frac{48 - 6}{2} = \frac{42}{2} = 21$$

Hence,
$$\frac{3}{8}(4)^2 + 15 = \frac{3(4)^2 - 6}{2}.$$

Again,
$$\frac{3}{8}(-4)^2 + 15 = \frac{48}{8} + 15 = 6 + 15 = 21$$

$$\frac{3(-4)^2 - 6}{2} = \frac{48 - 6}{2} = \frac{42}{2} = 21,$$

whence, $\frac{3}{8}(-4)^2 + 15 = \frac{3(-4)^2 - 6}{2}$

2. $\frac{5x^2}{11} - 3 + \frac{x^2}{2} = \frac{7x^2}{4} + \frac{9}{11} - 7. \quad x = \pm 2$

3. $2(x^2 - 10) - \frac{x^2}{5} = \frac{17}{25}x^2 + 12 - \frac{x^2 - 5}{4} + 1, \quad x = \pm 5.$

4. $x^2 - \frac{1}{9} - \frac{3x^2}{4} = \frac{5}{2}x^2 - 2x^2 - \frac{2}{9}. \quad x = \pm \frac{2}{3}.$

5. $4x^2 - 3(x^2 - 1) = \frac{7}{8}x^2 + \frac{9}{32} + 8x^2 - \frac{1}{4}.$

6. $\frac{5}{6} - x^2 + \frac{2x^2}{3} = \frac{8(x^2 - 1)}{10} - \frac{x^2 - 4}{7} + \frac{11}{6}.$

7. $\frac{x^2}{12} - \frac{1}{6} + \frac{3x^2}{7} = \frac{x^2 - 9}{4} - \frac{1}{12} + 15. \quad x = \pm 7.$

8. $4x^2 - 1 - \frac{x^2 - 31}{3} = 3x^2 + \frac{4x^2 - 21}{5} + \frac{1}{4}. \quad x = \pm 5\frac{1}{2}.$

9. $\frac{20}{2x} + x - \frac{10 + x^2}{x} = \frac{2x^2 + 19}{3x} - x + \frac{27}{x}. \quad x = \pm 10.$

10. $\frac{1}{3}x^2 - \frac{2}{5}(x^2 - 6) + 24 = \frac{5}{11}(3x^2 - 9) - x^2 + \frac{x^2 - 6}{2}.$

11. $\frac{m}{9m + 3x} + \frac{m}{6m - 2x} - \frac{9m^2 + mx}{54m^2 - 6x^2} = \frac{1}{5}.$

Clearing of fractions, we obtain, $30m^2 - 10mx + 45m^2 + 15mx - 45m^2 - 5mx = 54m^2 - 6x^2$. Uniting terms, $30m^2 = 54m^2 - 6x^2$. Collecting the known terms in the second member, and transposing $-6x^2$, we obtain, $6x^2 = 54m^2 - 30m^2$; $6x^2 = 24m^2$; whence $x^2 = 4m^2$ and $x = \pm 2m$.

12. $\frac{x+m}{x-n} - \frac{x-n}{2x-2m} = \frac{2nx-3m^2}{2(x^2-(m+n)x+mn)}.$

Clearing of fractions, $2x^2 - 2m^2 - x^2 + 2nx - n^2 = 2nx - 3m^2$.

Uniting terms, and $\left\{ \begin{array}{l} 2x^2 - x^2 = -m^2 + n^2 \therefore x^2 = n^2 - m^2; \\ \text{reducing;} \end{array} \right.$

whence $x = \pm \sqrt{n^2 - m^2}$.

$$13. \frac{x+7}{x-8} - \frac{x-8}{2x-14} = \frac{14x-282}{2x^2-30x+112}.$$

$$14. \frac{ax^2(b-a)}{m+ax} = m-ax. \quad x = \pm \frac{m}{\sqrt{ab}}.$$

$$15. \frac{x+a}{x-a} + \frac{x-a}{x+a} = 1 + \frac{12a^2}{x^2-a^2}. \quad x = \pm 3a.$$

$$16. \frac{x-a}{x+a} + \frac{x-b}{x+b} + \frac{x-c}{x+c} - 1 = 2 - 4 \frac{b(a+c)+ac}{(x+a)(x+b)(x+c)x}.$$

$$x = \pm \sqrt{\frac{-3abc}{a+b+c}}.$$

$$17. \frac{x}{a} + \frac{a}{x} = \frac{x}{b} + \frac{b}{x}. \quad x = \pm \sqrt{ab}.$$

383. To Deduce a Rule for solving a Complete Equation of the Second Degree.

Suppose the given equation reduced to the form $x^2+2px=q$ (1). If, now, we compare the first member of (1) with the square of $x+p$, we see that it differs from this square only in not having the term p^2 . In other words, if p^2 were added to the first member, this member would become a perfect square. Let us, then, add p^2 to the first member, and, to preserve the equality of the members, add the same quantity to the second member. Equation (1) then becomes $x^2+2px+p^2=q+p^2$ (2).

Extracting the square root of both members of (2), we obtain:

$$x+p = \pm \sqrt{q+p^2}, \text{ whence,}$$

$$x = -p \pm \sqrt{q+p^2}.$$

If, then, we substitute for x , in (1), either $-p+\sqrt{q+p^2}$, or $-p-\sqrt{q+p^2}$ the equation will be satisfied.

$$\text{Thus: } (-p+\sqrt{q+p^2})^2+2p(-p+\sqrt{q+p^2}) =$$

$$p^2-2p\sqrt{q+p^2}+q+p^2-2p^2+2p\sqrt{q+p^2} = q.$$

$$\text{Or, } (-p-\sqrt{q+p^2})^2+2p(-p-\sqrt{q+p^2}) =$$

$$p^2+2p\sqrt{q+p^2}+q+p^2-2p^2-2p\sqrt{q+p^2} = q.$$

In other words, the equation has *two roots*, $-p+\sqrt{q+p^2}$ and $-p-\sqrt{q+p^2}$.

As $x^2+2px=q$ may represent any equation of the second degree, we deduce the following

RULE.—Reduce the equation to the form $x^2+2px=q$. The first root equals half the coefficient of the first power of the unknown quantity, with its sign changed, plus the square root of the second member of the equation increased by the square of this half-coefficient. The second root equals half the coefficient of the first power of the unknown quantity, with its sign changed, diminished by the square root of the second member, plus the square of this half coefficient.

384. The process of adding p^2 to both members of (1), is called *Completing the Square*; and as p is one half the coefficient of x , we see that, in general,

The **Completion of the Square** consists in adding to both members of the given equation the square of one half the coefficient of the first power of the unknown quantity.

385. Corollary I.—The sum of the roots equals the coefficient of the first power of the unknown quantity, with its sign changed.

386. Corollary II.—The product of the roots equals the second member, with its sign changed.

The rule (383) should be learned so that the roots of any equation of the second degree may be written out at once, without the slightest hesitation, after the equation has been reduced to the form $x^2+2px=q$.

EXAMPLES.

1. Solve the equation $x^2-15x+56=0$.

Reducing to the form $x^2+2px=q$, we have $x^2-15x=-56$. Hence the first root is,

$$\begin{aligned} x &= \frac{15}{2} + \sqrt{-56 + \frac{225}{4}} = \frac{-224 + 225}{4} \\ &= \frac{15}{2} + \sqrt{\frac{1}{4}} = \frac{15}{2} + \frac{1}{2} = \frac{16}{2} = 8. \end{aligned}$$

and the second,

$$x = \frac{15}{2} - \sqrt{-56 + \frac{225}{4}} = \frac{-224 + 225}{4} = \frac{15}{2} - \frac{1}{2} = \frac{14}{2} = 7$$

and 8 and 7 are the required roots.

To verify: $(8)^2 - 15 \times 8 + 56 = 64 - 120 + 56 = 0.$

$(7)^2 - 15 \times 7 + 56 = 49 - 105 + 56 = 0.$

2. $x^2 - 12x + 32 = 0.$

$x = 8$ and $x = 4.$

3. $x^2 - 17x + 72 = 0.$

$\begin{cases} x = 9. \\ x = 8. \end{cases}$

4. $x^2 + 20x + 99 = 0.$

$\begin{cases} x = -11. \\ x = -9. \end{cases}$

5. $x^2 + 2x - 143 = 0.$

$\begin{cases} x = -13. \\ x = 11. \end{cases}$

6. $\frac{5x^2}{3} - 4x + \frac{4x^2}{9} = \frac{7x^2 - 21}{2} - \frac{4x^2 + 6}{7} - 3x + 1.$

Clearing of fractions,

$210x^2 - 504x + 56x^2 = 441x^2 - 1323 - 72x^2 - 108 - 378x + 126.$

Reducing to the required form, $x^2 + \frac{126}{103}x = \frac{1305}{103}.$ Then,

$$x = -\frac{63}{103} \pm \sqrt{\frac{1305}{103} + \frac{3969}{(103)^2}} = \frac{138384}{(103)^2} = -\frac{63}{103} \pm \frac{372}{103} = \frac{309}{103} \text{ or } -\frac{435}{103} = 3 \text{ or } -4\frac{23}{103}.$$

7. $x^2 - 23x + 132 = 0.$

(8.) $7x^2 - 5x = 48.$

9. $\frac{1}{2} + \frac{2x^2}{3} - \frac{x+3}{6} - \frac{1}{12} = \frac{14}{24} - \frac{x^2}{3} - \frac{1}{2}, \quad x = \frac{1}{2} \text{ or } -\frac{1}{3}.$

10. $2x^2 - 27x + \frac{x-3}{4} = \frac{7}{4}x^2 - \frac{51x}{2} - \frac{9}{4}, \quad x = 3 \text{ or } 2.$

11. $\frac{5}{7}x^2 + 2x - \left(\frac{1}{7}x^2 + 3x\right) = 2(x-30) - 3(x+1) + \frac{8x^2}{7}.$

12. $\frac{3x-2}{2x-5} + \frac{2x-5}{3x-2} = \frac{10}{3}, \quad x = \frac{13}{3} \text{ or } \frac{1}{7}.$

13. $\frac{x-1}{12} + \frac{13}{x} = 2, \quad x = 12 \text{ or } 13.$

$$14. \frac{x+6}{9} + \frac{x-15}{3} = \frac{x-12}{x}. \quad x=9 \text{ or } 3.$$

$$15. \frac{x+11}{11} + \frac{x+17}{17} = \frac{187-x^2}{187}. \quad x=-11 \text{ or } -17.$$

$$16. \frac{17-x^2}{5} - 12x - \frac{89x+7-3x^2}{11} = \frac{43x}{11} + \frac{42-51x^2}{55}.$$

$$17. \frac{19x^2-7}{2} + 12x - \frac{3(x^2-21)}{5} = \frac{13x^2+23}{3} - (x-7)4 - \frac{x^2+11}{2}.$$

$$18. \frac{x^2-3}{4} + \frac{4x^2-15}{4} - \frac{1}{2} = \frac{3x^2-10}{2} - (x^2+6) + x + 7.$$

$$19. x^2-1 + \frac{2x(2x+1)}{7} - 26 = x\left(x + \frac{1}{2} \cdot \frac{116-6x}{7}\right) - 18. \quad x=9 \text{ or } -1.$$

$$20. x-7 + \frac{x+6}{14} = \frac{6}{14-x} + \frac{x-6}{2}. \quad x=8 \text{ or } 12\frac{1}{4}.$$

$$21. \frac{x+\frac{1}{x}}{x-\frac{1}{x}} = \frac{12-4\left(1+\frac{2}{x}\right)}{4-\frac{4}{x}} \cdot \frac{1}{x-1} \quad x=1 \pm \sqrt{2}.$$

$$22. 3x + \frac{2x^2}{14} - (x+3)2 - 6 - \frac{5}{7}x = \frac{x^2+1}{5} - \frac{6x+8}{7} - \left(6 - \frac{2x-9}{35}\right).$$

$$23. \frac{8}{x+1} + \frac{5}{x+3} = \frac{7}{x+5} + \frac{1}{x^2+9x^2+23x+15}. \quad x=-2 \text{ or } -24.$$

$$24. \frac{x+3}{x-3} + \frac{x+6}{x-6} = -7. \quad x=5 \text{ or } 2.$$

$$25. \frac{x-7}{x-5} + \frac{x-8}{x-4} = \frac{14}{15} \quad x=10 \text{ or } 4\frac{5}{8}.$$

$$26. \frac{x-5}{x+5} + \frac{x+5}{x-5} = \frac{2x+51}{x+13}. \quad x=11 \text{ or } -7.$$

$$27. (x-5)\frac{x-6}{2} + \frac{x^2+1}{10} = \frac{x^2-2x}{7} + x^2-6x-6.$$

$$28. \frac{x^2}{2 + \frac{1}{3 + \frac{1}{4}}} + 13x = 75\frac{5}{6}. \quad x = -85 \text{ or } 5$$

$$29. x = \frac{1}{5 + \frac{1}{6 + \frac{1}{x}}}$$

$$30. (x-1)^4 + (x-2)^4 + (x-3)^4 + (x-4)^4 = 4(x-1)(x-2)(x-3)(x-4)$$

$$x = \frac{25 \pm \sqrt{-20}}{10}.$$

$$31. (x-1)^4 + (x-3)^4 + (x-5)^4 + (x-7)^4 = 4(x-1)(x-3)(x-5)(x-7)$$

$$x = \frac{20 \pm \sqrt{-20}}{5}.$$

$$32. \frac{3+2x}{2-x} - \frac{2-3x}{2+x} + \frac{16-x^2}{4} = \frac{1}{5-x} - \frac{1}{x-4} - \frac{5}{x^2-x-12}.$$

$$33. \frac{2x^{-1}-3}{3x^{-1}-5} - \frac{5}{2} = -\frac{30x^{-1}-50}{20x^{-1}-30}. \text{ Place } x^{-1}=y, \text{ i.e., substitute } y \text{ for } x^{-1}, \text{ and find, from the resulting equation, the value of } y \text{ (383); or consider } x^{-1} \text{ as the unknown quantity, and find (383) its values; thus, } x^{-1} = \frac{7}{4} \text{ or } 1, \text{ and hence } x = \frac{4}{7} \text{ or } 1; \text{ or, finally, } \frac{1}{x} \text{ may be substituted for } x^{-1} \text{ in the given equation; whence } \frac{1}{x} = \frac{7}{4} \text{ or } 1, \text{ and } x = \frac{4}{7} \text{ or } 1.$$

$$34. 24 + \frac{x^2}{3} - (x+1)5 = \frac{10x-x^2}{9} + x^2 - 8x - 18.$$

$$35. \frac{m-n}{x-n^2} \cdot \frac{1}{n} = \frac{m+n}{x+m^2} \cdot \frac{1}{m}. \quad x = mn \text{ or } -\frac{m^2+n^2}{m-n}.$$

$$36. mx^2 - \frac{mn}{m+l} = nx - lx^2.$$

$$\text{Transposing and factoring, } (m+l)x^2 - nx = \frac{mn}{m+l}.$$

Reducing to the form $x^2 + 2px = q$, $x^2 - \frac{n}{m+l}x = \frac{mn}{(m+l)^2}$

$$\therefore x = \frac{n}{2(m+l)} \pm \sqrt{\frac{nm}{(m+l)^2} + \frac{n^2}{4(m+l)^2} = \frac{4mn+n^2}{4(m+l)^2}}$$

$$= \frac{n \pm \sqrt{4mn+n^2}}{2(m+l)}.$$

$$37. \frac{mx}{b} + \frac{(b-2x)m}{x-m} = \frac{(x-2b)b}{x-m}. \quad x = \frac{b^2}{m} \text{ or } 2b+m.$$

$$38. \frac{x-a}{a-c} - \frac{c}{2a-x} = \frac{x-c}{2(a-c)} - \frac{x+c}{2a}. \quad x = 2a-c \text{ or } \frac{2a^2}{2a-c}.$$

$$39. \frac{bc-ba-3ac}{x^2-bx-cx+bc} + \frac{x-2a}{x-b} + \frac{3x-a}{x-c} + \frac{2(x+b)ax-cx-4}{x^2-(b+c)x+bc}$$

$$= 1 + \frac{2x}{x-c} + \frac{x+a}{x-b} + \frac{2(a-b)}{x^2-(b+c)x+bc}.$$

Clearing of fractions, transposing, and uniting terms,
 $2ax^2 - 4ax + 2abx = 4 - 2b + 2a$. Reducing to the form
 $x^2 + 2px = q$,

$$x^2 - (2-b)x = \frac{2-b+a}{a}.$$

$$\therefore x = \frac{2-b}{2} \pm \sqrt{\frac{2-b+a}{a} + \frac{4-4b+b^2}{4}}$$

$$= \frac{2-b}{2} \pm \frac{\sqrt{(4+4a)(-b+2)+ab^2}}{2\sqrt{a}}$$

$$40. \frac{x+2d}{3a-c} - \frac{2d+c}{x-3a} = 2. \quad x = 6a-c \text{ or } 3a-2d-c.$$

$$41. \frac{a}{a-x} + \frac{3a}{a+x} - \frac{2ax}{a^2-x^2} = \frac{a-x}{x} + \frac{16-5a^2}{ax+x^2}. \quad \text{The sum}$$

of the fractions in the first member is $\frac{4a}{a+x}$. The equation

may, then, be put under the form, $\frac{4a}{a+x} - \frac{a-x}{x} + \frac{5a^2-16}{ax+x^2} = 0$,

or, clearing of fractions, and reducing, $4a^2 + 4ax + x^2 = 16$,
 whence $x = \pm 4 - 2a$. It may frequently be convenient,
 as in this case, to change the order of the steps required
 by the rule (383).

$$42. \frac{a+x}{a-x} - \frac{a-x}{a+b} = \frac{b+x}{a-x}. \quad x=a \pm \sqrt{a^2-b^2}.$$

$$43. \frac{mx}{n} + \frac{nx}{m} = m^2 + \frac{mn^2}{x}.$$

$$44. \frac{1}{a-b-x} = \frac{1}{a} - \frac{1}{b} - \frac{1}{x}. \quad x=a \text{ or } -b.$$

45. $m^2+n^2-2nx+x^2-\frac{r^2x^2}{p^2}=0$. Transposing and factoring, $(p^2-r^2)x^2-2np^2x=-p^2(m^2+n^2)$. Reducing to the form

$$x^2+2px=q, \quad x^2-\frac{2np^2}{p^2-r^2}x=-p^2\frac{m^2+n^2}{p^2-r^2}; \text{ whence,}$$

$$\begin{aligned} x &= \frac{np^2}{p^2-r^2} \pm \sqrt{\left(-p^2\frac{m^2+n^2}{p^2-r^2} + \frac{n^2p^4}{(p^2-r^2)^2}\right)} \\ &= \frac{np^2}{p^2-r^2} \pm \frac{p}{p^2-r^2} \sqrt{r^2(m^2+n^2)-p^2m^2} \\ &= \frac{p}{p^2-r^2} [np \pm \sqrt{r^2(m^2+n^2)-p^2m^2}]. \end{aligned}$$

$$46. 4(a^2x^2+a^2c^2x+abd^2x)-9cd^2x^2+(ac^2+bd^2)^2=0.$$

$$x = -\frac{ac^2+bd^2}{2a+3d\sqrt{c}} \text{ or } -\frac{ac^2+bd^2}{2a-3d\sqrt{c}}.$$

$$47. (x-a)(x-b)+(x-b)(x-c)=(x-c)(x-a).$$

$$x=b \pm \sqrt{b^2-ab-bc+ac}.$$

$$48. \frac{a-x^{-1}}{b+x^{-1}} - \frac{b-x^{-1}}{a+x^{-1}} = \frac{a+b}{a-b}. \quad x = \frac{2}{\pm(a-b)\sqrt{5-(a+b)}}.$$

$$49. \frac{2x-m}{2(2x-k)} = \frac{k}{2x+m} - \frac{m^2}{4x^2-2kx+2mx-mk}.$$

$$x = \frac{k}{2} \pm \frac{1}{2} \sqrt{m^2+k^2-1}.$$

$$50. ax^2+ab=2ab^{\frac{1}{2}}x+bx^2.$$

$$x = \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}} \text{ or } \frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}.$$

$$51. \frac{(1+x)d^2+(1-x)c^2}{1-x} = \frac{cd}{x-x^2}.$$

$$x = \frac{c}{c+d} \text{ or } \frac{d}{c-d}.$$

$$52. x^2 + mx + dx + fm = f^2 - md + fn. \quad 53. \frac{3dy - 3y^2}{10d - 3y} = \frac{d}{6}.$$

$$54. \frac{2x^{-1} - b}{4x^{-1} - 2a} = \frac{a}{2x^{-1} + b} - \frac{b^2}{4x^{-2} - 2(a-b)x^{-1} - ab}.$$

$$55. \frac{a^2 + ax^{-1} + x^{-2}}{a + x^{-1}} + \frac{a^2 - ax^{-1} + x^{-2}}{a - x^{-1}} = \frac{ab}{3a - 4b + x^{-1}}.$$

$$x = -\frac{1}{3a} \text{ or } \frac{b}{3ab - 2a^2}.$$

$$56. \frac{amp}{x} - \frac{mp}{a} x = 1 - m^2 p^2 + ampb - bx.$$

$$x = amp \text{ or } \frac{a}{ab - mp}.$$

$$57. \frac{x+5m}{6m-k} - \frac{5m+k}{x+6m} = 1 - \frac{36m^2}{36m^2 - 6(k-x)m - kx}.$$

$$58. \frac{x+a}{x-a} + \frac{x+b}{x-b} = a.$$

$$59. \frac{x^2 - 3b - 2c}{b+c} - (a+b) = \frac{ax}{b+c} - x.$$

$$60. \frac{a-x}{a+x} - \frac{b+x}{a-x} = \frac{a-b}{a+b}. \quad 61. \frac{x-2b}{3a+c} - \frac{2b-c}{x-3a} = -1.$$

$$62. \frac{a+c(a+x)}{a+c(a-x)} + \frac{a+x}{x} = \frac{a}{a-2cx}. \quad x = -a \text{ or } \frac{a}{c} \cdot \frac{1+c}{3+2c}.$$

$$63. (7a^2 + 5m^2 + k^2)(x^2 + x + 1) = (7a^2 + 5k^2 + m^2)(x^2 - x + 1).$$

PROBLEMS.

1. Find a number such that three times its square, diminished by 26 times the number, shall equal 169.

Let x = the required number. Then, from the conditions,

$$3x^2 - 26x = 169; (1) \quad x^2 - \frac{26}{3}x = \frac{169}{3} \quad \text{Hence,}$$

$$x = \frac{13}{3} \pm \sqrt{\frac{169}{9} + \frac{169}{9}} = \frac{13}{3} \pm \sqrt{\frac{169}{9} + \frac{507}{9}} = \frac{676}{9} = \frac{13}{3} \pm \frac{26}{3}$$

$$= \frac{39}{3}, \text{ or } -\frac{13}{3} = 13, \text{ or } -4\frac{1}{3}$$

These roots both verify the equation: thus,

$$3(13)^2 - 26 \times 13 = 507 - 338 = 169$$

$$3\left(-\frac{13}{3}\right)^2 - 26\left(-\frac{13}{3}\right) = \frac{169}{3} + \frac{338}{3} = \frac{507}{3} = 169.$$

The first root evidently satisfies the conditions of the problem as stated, and so also the second root if we regard its sign.

It is to be observed that, if we had denoted the required number by $-x$, our equation would have been $3x^2 + 26x = 169$ (2), of which the roots are -13 and $4\frac{1}{3}$, differing from the roots of (1), only in the signs. This is precisely what we should anticipate, for either (1) or (2) may be changed into the other, by changing the sign of x .

2. A farmer bought a certain number of sheep for \$240. Had he paid the same sum for a number of sheep less by 3, each sheep would have cost \$4 more. Required, the number of sheep purchased.

Let x = the number of sheep purchased.

As x sheep cost \$240, one sheep cost $\frac{240}{x}$ dollars. If the number of sheep had been less by 3, it would have been $x-3$, and the cost of each sheep would have been $\frac{240}{x-3}$ dollars.

Hence, from the conditions, the cost of a sheep in the second case being greater by \$4 than in the first, $\frac{240}{x-3}$ is greater by four than $\frac{240}{x}$,

$$\text{or } \frac{240}{x-3} - 4 = \frac{240}{x} \quad (1). \quad \therefore \frac{240}{x-3} - \frac{240}{x} = 4, \quad 240x - 240x + 720 = 4x^2 - 12x$$

$$\text{or, } \frac{4x^2 - 12x - 720}{x^2 - 3x - 180}$$

$$x = \frac{3}{2} \pm \sqrt{\frac{720}{4} + \frac{9}{4}} = \frac{729}{4} = \frac{30}{2} \text{ or } -\frac{24}{2}$$

$$= 15 \text{ or } -12.$$

The root 15 satisfies the conditions of the problem as stated, for if 15 sheep cost \$240, one costs \$16, and if 12 sheep had been purchased for \$240, each would have cost \$20, i. e., the cost of each in the second case would have exceeded the cost in the first case by \$4.

But -12 will not satisfy the conditions of the problem, taken in its ordinary sense. We have found, however, that the negative sign always indicates the opposite of what is meant by the positive sign. We might, therefore, surmise that, as the positive result indicates the number of sheep bought, the negative result may refer to a number of sheep sold. This is, in fact, the case; the second root satisfies the conditions of the following problem:

A farmer sold sheep for \$240. Had he received the same sum for a number of sheep *greater* by 3, he would have received \$4 *less* for each sheep. Required the number of sheep.

If x denote the number of sheep sold, the statement of the problem reduces to $x^2 - 3x = 180$, and as this is the same equation given by the former problem, it necessarily gives the answers to both. Whenever the equation of a problem has two roots, if the problem does not admit of two solutions there is another problem, of which the statement gives the same equation, and the two roots of the equation are the answers to the two problems.

3. Find two numbers in the ratio of 7 to 12, the difference of whose squares is 380.

Let $7x$ be the first number; then will $12x$ be the second; and, from the conditions, $(12x)^2 - (7x)^2 = 380$; or, $144x^2 - 49x^2 = 380 \therefore 95x^2 = 380$; $x^2 = 4$, and $x = \pm 2$.

$7x = \pm 14$ and $12x = \pm 24$; $\therefore \pm 14$ and ± 24 are the numbers.

4. Find two numbers, such that the difference shall be to the greater as 5 is to 7, and the difference of their squares shall be 405.

Let $7x$ = the greater (arithmetically); then

$5x$ = the difference,

and $2x$ = the less. Hence, $(7x)^2 - (2x)^2 = 405$; or, $49x^2 - 4x^2 = 405$.

$45x^2 = 405$; $x^2 = 9 \therefore x = \pm 3$, and ± 21 and ± 6 are the numbers.

5. Find two numbers, such that the sum shall be to the less as b is to a , and the difference of their squares shall be $3a^2$.

$$\text{Ans. } \begin{cases} a^2 \sqrt{\frac{3}{b^2 - 2ab}}; \text{ the less.} \\ (ab - a^2) \sqrt{\frac{3}{b^2 - 2ab}}; \text{ the greater.} \end{cases}$$

6. A man bought sheep for \$ a . Had he purchased b more for the same sum, he would have paid \$ c less for each. Required, the number purchased.

Let x = the number purchased.

Then $\frac{a}{x}$ = the number of dollars paid for each.

Had he purchased b more for \$ a , $\frac{a}{x+b}$ would be the number of dollars paid for each.

$$\therefore c = \frac{a}{x}; \text{ whence, } x = -\frac{b}{2} \pm \frac{1}{2} \sqrt{\frac{4ab + b^2c}{c}}.$$

7. A and B set out at the same time from two places, and travel toward each other. When they meet, they find that A has traveled a miles more than B; A would have traveled B's journey in c days, and B would have traveled A's journey in b days. Required, the distances traveled by A and B respectively, and their distance apart at starting.

Let x = the number of miles traveled by A; then

$x - a$ = the number of miles traveled by B;

$\frac{x - a}{c}$ = the number of miles traveled by A in one day;

$\frac{x}{b}$ = the number of miles traveled by B in one day;

$\frac{x}{x - a} = \frac{cx}{x - a}$ = the number of days on which A traveled;

$\frac{x - a}{b} = \frac{b(x - a)}{x}$ = the number of days on which B traveled.

Since they started together, and traveled until they met,

$$\frac{cx}{x - a} = \frac{b(x - a)}{x} \therefore x = \frac{a(b \mp \sqrt{bc})}{b - c}; \quad x - a = \frac{a(c \mp \sqrt{bc})}{b - c}$$

$$\text{The whole distance is } \frac{a(b \mp 2\sqrt{bc} + c)}{b - c} = \frac{a(\sqrt{b} \pm \sqrt{c})^2}{\sqrt{b} \pm \sqrt{c}}$$

8. In the town of L—— the number of schools exceeds the number of churches by 20, and the product of the number of schools and the number of churches is 525. Required, the number of each. *Ans.* 35 schools; 15 churches.

9. On one of the public squares in L—— the whole number of trees is to the number of oaks as 7 is to 2; and if from the square of the number of elms we subtract the square of the number of oaks the remainder will be 336. Required, the number of each. *Ans.* 20 elms and 8 oaks.

10. An army corps leaves S——, and proceeds toward M——, and at the same time the troops of the enemy leave M——, and march toward S——. When the troops meet it is noticed that those from S—— have traveled 50 miles farther than their opponents. They would in 4 days have marched the distance their opponents have come, while it

would have taken the latter 9 days to perform the march accomplished by the troops from S——. Required, the distance from S—— to M——. *Ans.* 250 miles.

11. A detachment of the S—— troops was marching in column, showing 10 men more on each flank than at the head of the column. Had the number of men in front been increased by 1000, the detachment would have been drawn up in 10 lines. Required the number in the detachment. *Ans.* 11,000.

12. The population of Wilton is to the population of Salem as $\frac{1}{2}$ to $\frac{2}{3}$, and to the population of Merton as $\frac{1}{2}$ to $\frac{3}{4}$. The sum of the squares of the numbers indicating the population is 7,240,000. Required, the population of each town. *Ans.* 1200, 1600, and 1800, respectively.

13. A merchant, purchasing cloth, spent \$560.00, and the number of dollars paid for a yard was to the number of yards as 5 to 7. Required, the number of yards, and the price per yard. *Ans.* 28 yds. @ \$20 per yd.

14. A garden plot 18 metres long and 12 metres wide is to be surrounded by a gravel walk of uniform width, the area of the walk to be precisely equal to the area of the plot itself. Required the width of the walk. *Ans.* 3 M.

15. In Montebello Castle are two courts, each in the form of a square, one of which is paved with marble, and the other with granite. The second is 12 metres longer than the first, and they together contain 2664 square metres. Required, the width of each.

Ans. 42 and 30 metres, respectively.

16. A woman purchased a piece of silk, and one of velvet, 26 yds. in all, for \$356. Each piece cost as much per yd. as there were yds. in the piece. Required, the number of yards in each. *Ans.* 10 and 16.

17. The Crystal Springs Water Company has two reservoirs in Reims, the top of each being in the form of a

square. The area of the water-surface of the first is to the corresponding area of the second as a is to b . A side of the first is shorter by d than a side of the second. Required, the lengths of the sides.

Let x = a side of the first; then $x+d$ = a side of the second. The surface being, in each case, a square, x^2 and $(x+d)^2$ will be the respective areas. Hence, from the conditions, $x^2 : (x+d)^2 :: a : b$ or $x^2 : x^2 + 2xd + d^2 :: a : b$. The product of the means being equal to the product of the extremes, $bx^2 = ax^2 + 2adx + ad^2$;

$$\text{whence } (b-a)x^2 - 2adx = ad^2; \quad \text{or,} \quad x^2 - \frac{2ad}{b-a}x = \frac{ad^2}{b-a}$$

$$\begin{aligned} x &= \frac{ad}{b-a} \pm \sqrt{\frac{ad^2}{b-a} + \frac{a^2d^2}{(b-a)^2}} = \frac{bad^2 - a^2d^2 + a^2d^2}{(b-a)^2} \\ &= \frac{ad}{b-a} \pm \frac{d\sqrt{ab}}{b-a} = \frac{a \pm \sqrt{ab}}{(b-a)}d = \pm \frac{\sqrt{a}(\sqrt{b} \pm \sqrt{a})}{(\sqrt{b})^2 - (\sqrt{a})^2}d \end{aligned}$$

Reducing the fraction to its simplest form, by striking out the common factor $\sqrt{b} + \sqrt{a}$, $x = \pm \frac{\sqrt{a}}{\sqrt{b} - \sqrt{a}}d$. Taking the positive value,

$$\begin{aligned} x+d &= \frac{\sqrt{a} + \sqrt{b} - \sqrt{a}}{\sqrt{b} - \sqrt{a}}d, \text{ since } \frac{d\sqrt{a}}{\sqrt{b} - \sqrt{a}} + d = d \frac{(\sqrt{a} + \sqrt{b} - \sqrt{a})}{\sqrt{b} - \sqrt{a}} \\ &= \frac{\sqrt{b}}{\sqrt{b} - \sqrt{a}}d. \text{ Hence the required lengths are } \frac{\sqrt{a}}{\sqrt{b} - \sqrt{a}}d, \text{ and } \\ &\frac{\sqrt{b}}{\sqrt{b} - \sqrt{a}}d. \text{ The negative values are answers to another problem, of} \\ &\text{which the enunciation gives rise to the same equation used here.} \end{aligned}$$

18. A gentleman having 175 deer, wished to put them in a certain number of pastures, putting the same number of deer in each. Before having his intentions carried out, however, he determined to reserve two of the pastures for other purposes. He was then obliged to put, in each pasture, 10 more deer than he would have put there originally. Required, the number of pastures that were to have been used at first.

Ans. 7.

19. Three bins are filled with corn. It would require m bushels to fill the first as many times as there are bushels in the second, and d bushels to fill the first as many times

as there are bushels in the third. The sum of the squares of the number of bushels in the second and the number in the third, is c . Required, the contents of the bins.

Let x = the number of bushels in the first. Then, since m bushels would suffice to fill the first as many times as there are bushels in the second, and m bushels would suffice to fill the first $\frac{m}{x}$ times,

$$\frac{m}{x} = \text{the number of bushels in the second.}$$

Similarly, $\frac{d}{x}$ = the number of bushels in the third. Hence, from

the conditions, $\frac{m^2}{x^2} + \frac{d^2}{x^2} = c$ (1), or $m^2 + d^2 = cx^2$; whence,

$$x = \sqrt{\frac{m^2 + d^2}{c}}$$

$$\frac{m}{x} = \frac{m}{\sqrt{\frac{m^2 + d^2}{c}}} = \frac{m}{\sqrt{m^2 + d^2}} \cdot \frac{\sqrt{c}}{\sqrt{c}} = \frac{m\sqrt{c}}{\sqrt{m^2 + d^2}} = m \sqrt{\frac{c}{m^2 + d^2}}$$

$$\frac{d}{x} = \frac{d}{\sqrt{\frac{m^2 + d^2}{c}}} = \frac{d\sqrt{c}}{\sqrt{m^2 + d^2}} = d \sqrt{\frac{c}{m^2 + d^2}}$$

Hence, $\sqrt{\frac{m^2 + d^2}{c}}$, $m \sqrt{\frac{c}{m^2 + d^2}}$, and $d \sqrt{\frac{c}{m^2 + d^2}}$, are the respective contents.

20. A man has 3 purses. The product of the number of dollars in the first and the number in the second, is 63; the product of the number in the first and the number in the third, is 77. The square of the number in the third, increased by the square of the number in the second, is 202. Required, the contents of the different purses.

Ans. \$7, \$9, \$11.

21. A commission merchant receives an order to purchase 18 Myr. G. of sugar, if it can be purchased at the rate of \$18 per Myr. G. If, however, the sugar should be dearer

or cheaper, he is to buy as many Myr. G. less or more than 18, as each costs more or less than \$18. In filling the order he expended \$315. How much sugar did he purchase?

Ans. 15 Myr. G., or 21 Myr. G.

22. A railway train left Miltown for Danville, and traveled a miles per hour. After it had traveled b miles, the express, whose regular time between the two cities was n hours, left Danville for Miltown. After the express had traveled as many hours as it went miles in one hour, it passed the Miltown train, on time. Required, the distance from Miltown to Danville.

Ans. $\frac{n}{2}(n-a \pm \sqrt{(n-a)^2 - 4b})$.

RADICAL EQUATIONS.

387. If, when an equation has been so transformed as to be free from radicals, it can be reduced to the form $x^2 + 2px = q$, it may, of course, be solved at once by the rule already given. The directions previously laid down, for clearing an equation of radicals, will, with the rule just referred to, enable us to solve any radical equation of the second degree. The work, however, may frequently be shortened by the use of special expedients, such as rationalizing the terms of fractions, etc. The student must rely on his own skill and ingenuity to determine the best course to pursue in any particular case.

EXAMPLES.

1. Let it be required to solve the equation

$$\sqrt{2x-7} - 2\sqrt{3x-8} + 5 = 6. \quad (1)$$

Transposing, $\sqrt{2x-7} - 2\sqrt{3x-8} = 1. \quad (2)$

Squaring both members of (2), we obtain $2x-7-2\sqrt{3x-8}=1$ (3); $2x-8=2\sqrt{3x-8}$; or, $\sqrt{3x-8}=x-4. \quad (4)$

Squaring both members of (4), we obtain $3x-8=x^2-8x+16$; reducing,

$$x^2 - 11x = -24 \quad x = \frac{11}{2} \pm \sqrt{-\frac{96}{4} + \frac{121}{4}} = \frac{11}{2} \pm \frac{5}{2} = 8 \text{ or } 3.$$

To verify the first value:

$$\sqrt{16-7-2\sqrt{24-8}}=\sqrt{9-2\sqrt{16}}=\sqrt{9-8}=\sqrt{1}=1.$$

To verify the second, we must note the double sign, that is to be understood before every radical of an even degree, as understood before the second radical in the given equation. Then, taking the negative sign,

$$\sqrt{6-7-2(-1\sqrt{9-8})}=\sqrt{-1+2}=+1=1.$$

$$2. \sqrt{3x-1}+\sqrt{3x+1}=2. \quad x=1 \text{ or } \frac{8}{3}.$$

$$3. 2\sqrt{x+6}-\sqrt{x-1}=6. \text{ Verify.}$$

$$4. 1+4\sqrt{4x+1}=\frac{\sqrt{(4x+1)^3}-2}{x}. \text{ Clearing of fractions,}$$

$$x+4x\sqrt{4x+1}=\sqrt{(4x+1)^3}-2; \sqrt{(4x+1)^3}=(4x+1)\sqrt{4x+1}.$$

Hence, $x+4x\sqrt{4x+1}=4x\sqrt{4x+1}+\sqrt{4x+1}-2$. Uniting terms, $x+2=\sqrt{4x+1}$; whence, squaring,

$$x^2+4x+4=4x+1; \quad x^2=-3 \quad x=\pm\sqrt{-3}.$$

$$5. \sqrt{\frac{3}{2}}+x-\sqrt{4+x}-\sqrt{1-x}=0. \quad x=\frac{1}{2} \text{ or } -\frac{3}{2}.$$

$$6. \sqrt{\frac{3m^3}{x^2}+m^2}-3=m+1-\sqrt{\frac{3m^3}{x^2}-2}. \quad x=\pm m.$$

$$7. \sqrt{1+x}-\sqrt{ab}=1-\sqrt{ab-x}. \quad x=ab-1 \text{ or } 0.$$

$$8. \frac{\sqrt{x}}{21-\sqrt{x}}+\frac{21-\sqrt{x}}{\sqrt{x}}=\frac{5}{2}. \quad x=49 \text{ or } 196.$$

$$9. \frac{1}{a-\sqrt{a^2-x^2}}-\frac{1}{a+\sqrt{a^2-x^2}}=\frac{a}{x^2} \quad x=\pm\frac{a\sqrt{3}}{2}.$$

$$10. \frac{x}{\sqrt{a^2+x^2}-x}=b. \quad x=\pm\frac{ab}{\sqrt{2b+1}}.$$

$$11. \frac{\sqrt{1+x}}{1+\sqrt{1+x}}=\frac{\sqrt{1-x}}{1-\sqrt{1-x}}. \quad x=\pm\frac{\sqrt{3}}{2}.$$

$$12. \frac{\sqrt{x^2+a}-\sqrt{x^2-a}}{\sqrt{x^2+a}+\sqrt{x^2-a}}=\frac{a}{2}.$$

Multiplying both terms of the first member by the numerator, we obtain,

$$\frac{(\sqrt{x^2+a}-\sqrt{x^2-a})^2}{2a} = \frac{a}{2}; \quad (\sqrt{x^2+a}-\sqrt{x^2-a})^2 = a^2;$$

whence,

$$\sqrt{x^2+a}-\sqrt{x^2-a} = \pm a,$$

Transposing, $\sqrt{x^2+a} = \pm a + \sqrt{x^2-a}$.

Squaring, $x^2+a = a^2 \pm 2a\sqrt{x^2-a} + x^2-a$, or

$$2a-a^2 = \pm 2a\sqrt{x^2-a}; \text{ dividing both}$$

members by $2a$, $1-\frac{a}{2} = \pm\sqrt{x^2-a}$; whence

$$1-a+\frac{a^2}{4} = x^2-a, \text{ or } 1+\frac{a^2}{4} = x^2.$$

$$\therefore x = \pm\sqrt{\frac{a^2}{4}+1} = \pm\sqrt{\frac{a^2+4}{4}} = \pm\frac{\sqrt{a^2+4}}{2}.$$

$$13. \frac{\sqrt{x^2+6}-\sqrt{x^2-6}}{\sqrt{x^2+6}+\sqrt{x^2-6}} = \frac{1}{3}. \quad x = \pm\sqrt{10}.$$

$$14. \frac{\sqrt{a^2+x^2}+x}{\sqrt{a^2+x^2}-x} = \frac{m}{n}. \quad x = \pm \frac{a(m-n)}{2\sqrt{mn}}.$$

$$15. \frac{\sqrt{c+x}+\sqrt{c-x}}{\sqrt{x}} = \sqrt{\frac{x}{m}}. \quad x = \pm 2\sqrt{cm-m^2}.$$

$$16. \sqrt{\frac{x+a}{x}} + 2\sqrt{\frac{a}{x+a}} = b^2\sqrt{\frac{x}{x+a}} \quad (1). \text{ Multiplying both}$$

members of (1) by $\sqrt{\frac{x+a}{x}}$, we obtain $\frac{x+a}{x} + 2\sqrt{\frac{a}{x}} = b^2$;

or $1 + \frac{a}{x} + 2\sqrt{\frac{a}{x}} = b^2$ (2). Arranging the terms of (2),

$1 + 2\sqrt{\frac{a}{x}} + \frac{a}{x} = b^2$ (3). Extracting the square root of both members of (3):

$1 + \sqrt{\frac{a}{x}} = \pm b$; whence, $1 \mp b = -\sqrt{\frac{a}{x}}$, and squaring,

$$(1 \mp b)^2 = \frac{a}{x}; \text{ or } x(1 \mp b)^2 = a \quad \therefore x = \frac{a}{(1 \mp b)^2}.$$

$$17. \frac{3+\sqrt{9-x^2}}{3-\sqrt{9-x^2}} = 25. \quad x = \pm \frac{15}{13}.$$

$$18. \frac{d-\sqrt{2dy-y^2}}{d+\sqrt{2dy-y^2}} = \frac{y}{d-y}.$$

$$19. \frac{m+x+\sqrt{2mx+x^2}}{m+x} = c \text{ (1).} \quad \text{Dividing, as indicated in}$$

the first member, we obtain,

$$1 + \frac{\sqrt{2mx+x^2}}{m+x} = c \text{ (2); whence, } \sqrt{\frac{2mx+x^2}{(m+x)^2}} = c-1 \text{ (3),}$$

and, squaring both members of (3),

$$2mx+x^2 = (c^2-2c+1)(m+x)^2 = (c^2-2c)(m+x)^2 + (m+x)^2$$

$$2mx+x^2 = (c^2-2c)(m+x)^2 + m^2 + 2mx + x^2.$$

Uniting terms, and transposing,

$$(2c-c^2)(m+x)^2 = m^2; \text{ whence, } (m+x)^2 = \frac{m^2}{2c-c^2},$$

$$m+x = \pm \frac{m}{\sqrt{2c-c^2}}; \quad x = -m \pm \frac{m}{\sqrt{2c-c^2}} = \frac{m(1 \pm \sqrt{2c-c^2})}{\sqrt{2c-c^2}}$$

$$20. \quad m - \sqrt{c^2+x^2} = \frac{c^2-x^2}{c+\sqrt{c^2+x^2}}. \quad x = \frac{\pm c^{\frac{3}{2}}\sqrt{3c-2m}}{c-m}.$$

$$21. \quad \sqrt{\frac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}}} = x-2.$$

$$\text{Squaring, } \frac{x+\sqrt{x^2-9}}{x-\sqrt{x^2-9}} = (x-2)^2; \text{ rationalizing the denom-}$$

$$\text{inator in the first member, } \frac{(x+\sqrt{x^2-9})^2}{9} = (x-2)^2, \text{ whence,}$$

$$\frac{x+\sqrt{x^2-9}}{3} = x-2; \quad x+\sqrt{x^2-9} = 3x-6, \text{ or } \sqrt{x^2-9} = 2(x-3),$$

$$\therefore x^2-9 = 4(x-3)(x-3) \therefore x+3 = 4(x-3) \text{ or, } 9-x^2 = 4(x-3)^2$$

$$\therefore 4(3-x) = -(3+x) \therefore x=5. \text{ Or, we may solve directly the equation } x^2-9 = 4x^2-24x+36, \text{ whence, } x=5 \text{ or } 3.$$

22. $\sqrt[3]{\frac{1}{1+x}\left(\frac{1}{1+x}\right)^{\frac{1}{2}}} = \frac{\sqrt{2x}}{12}$; this may be written
 $\sqrt[3]{\left(\frac{1}{1+x}\right)^{\frac{3}{2}}} = \frac{\sqrt{2x}}{12}$, or $\left(\frac{1}{1+x}\right)^{\frac{1}{2}} = \frac{\sqrt{2x}}{12}$; whence, $x^2+x=72$

GENERAL PROPERTIES

OF EQUATIONS OF THE SECOND DEGREE.

As every equation of the second degree may be reduced to the form $x^2+2px=q$ (1), p and q being numerical or literal, positive or negative, entire or fractional, (1) may represent any equation whatever of the second degree.

388. The General Properties of the Equation of the Second Degree are five, viz:—

I. Every equation of the second degree has two roots, and only two.

Taking $x^2+2px=q$ (1), and completing the square (381), we obtain, $x^2+2px+p^2=q+p^2$ (2); or, $(x+p)^2=(\sqrt{q+p^2})^2$ (3); any quantity being the square of its own square root. Transposing in (3), we have $(x+p)^2-(\sqrt{q+p^2})^2=0$ (4). The first member of (4), being the difference of the squares of two quantities, is the product of the sum and difference of these quantities. Hence, factoring, we have,

$$(x+p+\sqrt{q+p^2})(x+p-\sqrt{q+p^2})=0 \quad (5).$$

Equation (5) may be satisfied by making either factor of the first member equal to zero, the other factor being any quantity >0 or <0 , since the product of zero by any quantity is zero; and the equation can be satisfied in no other way.

Placing $x+p-\sqrt{q+p^2}=0$, we have $x=-p+\sqrt{q+p^2}$.

Placing the other factor equal to zero, we obtain

$$x+p+\sqrt{q+p^2}=0; \text{ whence } x=-p-\sqrt{q+p^2}.$$

Either of these values of x will, if substituted in (5), satisfy the equation, and no other values of x will satisfy this equation, for the product of two quantities cannot be zero, unless one of the factors is zero, and neither of the factors in the first member of (5) can be zero, unless x has one of the two

values just given. As each value of x is a root of the equation (5), it will be a root of (1), from which (5) was derived. In other words, the equation (1) has two roots, and only two.

This may also be shown in another way. That the equation has always two roots, $-p+\sqrt{q+p^2}$ and $-p-\sqrt{q+p^2}$, appears from (383). We are to show that it can have no more. Suppose that it has three roots, a , b , and c . Since every root of an equation must, when substituted for the unknown quantity, satisfy the equation, we have

$$\left. \begin{array}{l} (6) a^2+2pa=q \\ (7) b^2+2pb=q \\ (8) c^2+2pc=q \end{array} \right\} \begin{array}{l} \text{Subtracting (7) from (6), member from} \\ \text{member, we have } a^2-b^2+2p(a-b)=0 \text{ (9);} \\ \text{or, dividing each member of (9) by } a-b, \\ a+b+2p=0 \text{ (10). Similarly, from (8) and (7), we find} \\ b+c+2p=0 \text{ (11). Subtracting (11) from (10), member from} \\ \text{member, we obtain } a-c=0, \text{ or } a=c, \text{ i. e., the supposed} \\ \text{third root is identical with the first. In a similar manner} \\ \text{we could show that any quantity, assumed as a root of the} \\ \text{equation, must be identical with either (a or b) one of the} \\ \text{roots obtained as in (383), i. e., the equation cannot have} \\ \text{more than two roots.} \end{array}$$

II. *The first member of every equation of the form $x^2+2px-q=0$, may be resolved into two binomial factors, of the first degree with respect to the unknown quantity, one the unknown quantity minus the first root, and the other the unknown quantity minus the second root.*

If we denote $-p+\sqrt{q+p^2}$ by x' and $-p-\sqrt{q+p^2}$ by x'' , $x+p-\sqrt{q+p^2}$ will be denoted by $x-x'$, and $x+p+\sqrt{q+p^2}$ by $x-x''$; hence equation (5) of (I) will become $(x-x')(x-x'')=0$, in which the first member is evidently factored as indicated in the proposition.

It follows from what has just been proved that, if we know the roots of an equation we may find the equation itself by subtracting them in succession from the unknown quantity, and then multiplying the resulting binomials together by (76), or, which is the same thing, we may write

out the equation at once by the use of **III** and **IV**. Thus, if the roots of an equation are $5+\sqrt{-12}$ and $5-\sqrt{-12}$, we may form the binomial factors, $x-5-\sqrt{-12}$ and $x-5+\sqrt{-12}$, and write out the product, (76) $x^2-10x+37$, which will be the first member of the equation, 0 being the second. Hence the equation is $x^2-10x+37=0$. The proposition just demonstrated may also be expressed thus:—

If a is a root of the reduced equation, $x^2+2px-q=0$, then is the first number of this equation exactly divisible by $x-a$.

It also appears, from what has already been said, that the converse of this is true, i. e.,

If the polynomial $x^2+2px-q$ is exactly divisible by $x-a$, then is a a root of the equation $x^2+2px-q=0$.

The applicability of these principles to the theory of Factoring is obvious and interesting. In order to determine whether a factor of the form $x-a$ enters a given polynomial, we have only to see if the substitution of a for x will reduce the given polynomial to zero.

III. *The algebraic sum of the roots of an equation of the second degree is equal to the coefficient of the first power of the unknown quantity, with its sign changed.*

See (381-5).

IV. *The product of the two roots of an equation of the second degree, equals the second member with its sign changed, i. e., equals the absolute term.*

See (381-6).

By the use of **II**, **III** and **IV**, we are enabled to write out, at once, any equation of which the roots are given.

EXAMPLES.

1. Find the equation of which the roots are -7 and $+3$.

The algebraic sum of the roots is -4 . Hence the coefficient of x is $+4$.

The product of the roots is -21 . Hence the absolute term is -21 , or the second member, if the completely known

terms be placed therein, is $+21$. The equation, then, is $x^2+4x-21=0$, or $x^2+4x=21$.

The Roots are:	Required, the Equation.
2. -12 and $+5$.	
3. -7 and -8 .	<i>Ans.</i> $x^2+15x+56=0$.
4. $+19$ and -1 .	<i>Ans.</i> $x^2-18x-19=0$.
5. $+\sqrt{-5}$ and $-\sqrt{-5}$.	<i>Ans.</i> $x^2+5=0$.
6. $2+\sqrt{-7}$ and $2-\sqrt{-7}$.	
7. $3\pm\sqrt{-42}$.	<i>Ans.</i> $x^2-6x+51=0$.
8. $7\pm\sqrt{19}$.	<i>Ans.</i> $x^2-14x+30=0$.
9. $-5\pm\sqrt{21}$.	<i>Ans.</i> $x^2+10x+4=0$.
10. $-11\pm\sqrt{-13}$.	

Find the binomial factors of the first members of the following equations:

1. $x^2-4x+4=0$.	4. $x^2-29x+108=0$.
2. $x^2-7x+12=0$.	5. $x^2+13x-109=0$.
3. $x^2-14x-27=0$.	6. $x^2-x+7=0$.
7. $2x^2+x-6=0$.	<i>Ans.</i> $2(x+2)(x-\frac{3}{2})$.
8. $3x^2-10x-25=0$.	<i>Ans.</i> $(x-5)3(x+\frac{5}{3})$.

The process, just illustrated, of building up equations from their roots, is known as the **Composition of Equations**.

V. *If the second member (q) of an equation of the second degree be negative, it cannot be numerically greater than the square of half the coefficient of the first power of the unknown quantity, if the roots be real.*

To prove this we shall first show that, the sum of two quantities being constant, their product will be the greatest possible when they are equal. Suppose $2p$ is the sum of the quantities, and let $2d$ represent their difference; then

$p+d$ = the greater quantity, and $p-d$ the less. If q represent their product, we shall have,

$$(p+d)(p-d)=p^2-d^2=q \quad (1).$$

Now, the less we subtract from p^2 , the greater must be the result; i. e., the less the value of d , the greater will be the value of q . Hence, the product will be the greatest possible when $d=0$; when the quantities become equal, and the product becomes p^2 . It is plain that the greatest value of the product would be p^2 , even if p were negative, for p^2 would always be positive.

We see, then, that if the sum of two quantities is numerically $2p$, the greatest value of the product of those quantities is p^2 . Now in the equation of the second degree, the sum of the roots is numerically $2p$, and, if the second member be negative, the product is $-q$. Hence, the greatest value that the product of the roots can have, i. e., the greatest numerical value that q may have, is p^2 .

That no limit can be put on the numerical value of the second member if positive, appears from the fact that if we consider the product in (1) to be negative, (as it would be if the second member of $x^2+2px=q$ were positive), it may increase numerically without limit, as long as d increases.

DISCUSSION OF THE FOUR FORMS.

389. An Arbitrary Quantity is a quantity to which we may assign values at pleasure.

390. To Discuss an equation or problem is to make all possible suppositions on the arbitrary quantities therein, and to explain the results.

391. In the general equation $x^2+2px=q$, if p and q are not both positive, or both negative, one is positive and the other negative. Hence, every equation of the second degree must be expressed by one of the **Four Forms**:

$$\begin{array}{ll} x^2+2px = q & \text{I.} \\ x^2-2px = q & \text{II.} \\ x^2+2px = -q & \text{III.} \\ x^2-2px = -q & \text{IV.} \end{array}$$

Solving these equations, we obtain the roots:

$$x = -p \pm \sqrt{q + p^2} \quad (1)$$

$$x = +p \pm \sqrt{q + p^2} \quad (2)$$

$$x = -p \pm \sqrt{-q + p^2} \quad (3)$$

$$x = +p \pm \sqrt{-q + p^2} \quad (4)$$

Under these forms (1), (2), (3), (4), must, then, be expressed the roots of every equation of the second degree.

We shall proceed to the discussion of:

A. The Four Forms themselves, and

B. The Roots of the Four Forms.

A. DISCUSSION OF THE FOUR FORMS THEMSELVES.

This discussion is based on the *General Properties* of the equation of the second degree. (388.)

I. In the first form the product of the roots (388, IV) is $-q$; hence they have contrary signs (62). Their sum (388, III) is $-2p$; hence (43) the negative root is numerically the greater.

II. In the second form the product of the roots is $-q$; hence they have contrary signs. Their sum is $+2p$; therefore the positive root is numerically the greater.

III. In the third form the product of the roots is $+q$ (388, IV); hence they have the same sign (388, III). Their sum is $-2p$; therefore they are both negative.

IV. In the fourth form the product of the roots (388, IV), is $+q$; hence they have the same sign. Their sum being (388, III) $+2p$, they are both positive.

Suppose $q=0$ and $p>0$. Then,

I. The product of the roots in the first form is zero; hence one of the roots is zero. Their sum is $-2p$; therefore the other root is $-2p$.

II. In the second form the product of the roots is 0; hence one of them is 0. Their sum is $+2p$, and therefore the other root is $+2p$.

III. In the third form the product of the roots is 0, and their sum is $-2p$. Hence one root is 0, and the other is $-2p$. The third form, then, under the hypothesis $q=0$, becomes the same as the first.

IV. When $q=0$, the fourth form becomes identical with the second.

It is usual to disregard the sign of zero, as this sign is unimportant. This is not always the case, but the exceptions will be noted at the proper time.

Again, suppose $p=0$, and q is not zero.

I. In the first form the product of the roots is $-q$; hence they have unlike signs. Their sum is 0; hence they are numerically equal, as, in adding, one precisely destroys the other.

II. In the second form, the product being $-q$, the signs of the roots are unlike, and the sum being zero, the roots are numerically equal. This form is, then, identical with **I**.

III and **IV.** In these forms the product of the roots is $+q$; hence the roots have the same sign. The product, q , not being zero, neither root can be zero. The sum of the roots is 0. Hence we have, in each form, two quantities having the same sign (and neither of which is zero), whose sum is zero. But there can be no such quantities, since a sum equal to zero can be obtained only in one of two ways, viz: 1°. Adding quantities, each of which is zero. 2°. Adding a positive quantity to a numerically equal negative one.

It is impossible, then, to find quantities that will satisfy the equations **III** and **IV**. But since this is the case, there must be something impossible or absurd involved in our supposition. Now if p is 0, $p^2=0$, and q , being greater than zero, is greater than p^2 . Hence our supposition that $p=0$, $q>0$ conflicts with (388, **V**), an essential property of equations of the second degree, and we should therefore expect it to lead to some impossibility in satisfying the equation.

It should be observed that when $p=0$ the equations are of the class called *Incomplete*.

Finally, suppose $p=0$ and $q=0$.

Under this hypothesis the product of the roots, in each form, is 0; hence one root in each is 0. The sum of the roots is 0; hence both roots in each form are 0.

B. ROOTS OF THE FOUR FORMS.

$$x = -p \pm \sqrt{q+p^2} \quad (1) \qquad x = -p \pm \sqrt{-q+p^2} \quad (3)$$

$$x = +p \pm \sqrt{q+p^2} \quad (2) \qquad x = +p \pm \sqrt{-q+p^2} \quad (4)$$

As, in obtaining the four forms, we have made every possible supposition on the signs of p and q , we shall now speak of their numerical values only. Neither p nor q can be less than 0, in any form, for the only effect of such a supposition would be to change the sign of p or q , and therefore to change this form into one of the others.

Since $q+p^2 > p^2$, $\sqrt{q+p^2} > \sqrt{p^2}$; i. e., $\sqrt{q+p^2} > p$.

Since $p^2 - q < p^2$, $\sqrt{p^2 - q} < \sqrt{p^2}$; i. e., $\sqrt{p^2 - q} < p$, and as, in adding quantities having different signs, we always prefix to the result the sign of the greater quantity, the signs of the roots in (1) and (2) will be the same as the signs of the radical parts, and in (3) and (4) the same as the signs of the rational parts. Again, in each of the first two forms the quantity under the radical sign is positive, for p^2 is positive whatever the sign of p , and q is not negative in either (1) or (2). The roots in the first two forms will, then, always be real, whatever the relative values of p and q .

(1.) In the first form, as the radical gives sign to the root, the first root will be positive, and the second negative. The roots in this form may be expressed thus:

$$\left. \begin{array}{l} \text{First Root, } -(p - \sqrt{q+p^2}) \\ \text{Second Root, } -(p + \sqrt{q+p^2}) \end{array} \right\} \text{ (388 I.)}$$

As p and $\sqrt{q+p^2}$ are here both positive, their sum is numerically greater than their difference. Hence, the second root in this form will be numerically greater than the first root.

(2.) In the second form, as the radical gives sign to the root, the first root will be positive, and the second negative.

The positive root being the arithmetical sum of p and $\sqrt{q+p^2}$, will be numerically greater than the negative root, which is the arithmetical difference of the same quantities.

(3). In the third form, as the rational part gives sign to the root, both roots will be negative. Expressing the

roots thus: $\left\{ \begin{array}{l} -(p - \sqrt{-q+p^2}) \\ -(p + \sqrt{-q+p^2}) \end{array} \right\}$ we see that the second

root is numerically the greater. The sign of the quantity under the radical sign will evidently depend on the relative values of p^2 and q . Now p^2 may be equal to q , or it may be either greater or less than q . If $p^2 > q$, $p^2 - q$ is positive, and the roots are both real. If $p^2 = q$, the radical part of each root reduces to 0, and both roots reduce to $-p$; the equation has, then, under this supposition, two equal roots. If $p^2 < q$, $p^2 - q$ is negative; and hence the radical part of each root becomes imaginary. Now, an imaginary quantity is one that, in the ordinary sense of the term, has no existence, and if such be the character of the roots under this supposition, we must examine carefully into the hypothesis to see if any impossibility or absurdity is involved.

We have found (388, V) that, when the second member is negative, it cannot be greater, numerically, than the square of one half of the coefficient of the first power of the unknown quantity. Hence the supposition, in (3) or (4), that $p^2 < q$, is absurd, and we should expect the values of the roots corresponding to this supposition to be impossible or imaginary. Whenever a supposition leads to imaginary results, we interpret these results as indicating that the supposition is absurd.

(4). In the fourth form, the rational parts giving signs to the roots, both roots are positive, and the first is evidently the greater, numerically. If $p^2 = q$, the two roots are equal, each being $+p$. If $p^2 > q$ the quantity under the radical sign is, in each case, positive, and both roots are real. If $p^2 < q$, $p^2 - q$ is negative, and both roots are imaginary. These imaginary roots result from the same absurdity be-

fore pointed out, viz., the supposition that in (4) p^2 can be less than q .

It will be noticed that the roots of an equation of the second degree can never be imaginary, unless the second member of the given equation (when in the reduced form) is negative. It will be seen also, that if one root is imaginary, both must be so.

We might also remark, in this connection, that if one of the roots is irrational, both must be irrational.

Suppose $p=0$, and q is not zero.

(1) and (2). The first and second forms become, under this hypothesis, the same. The first root in each becomes $+\sqrt{q}$, and the second $-\sqrt{q}$; i. e., the roots in each form are numerically equal, and have contrary signs.

(3) and (4). These become identical. The first root reduces to $+\sqrt{-q}$, and the second to $-\sqrt{-q}$. These roots being imaginary, there must be some absurdity or impossibility involved in the supposition. Examining the hypothesis, we see that it is merely one form of the supposition $p^2 < q$, for if $p=0$, $p^2=0$, and if q is not zero, $p^2 < q$, which, as we have before found, is absurd. We should therefore expect the roots to be imaginary, both being so if one is so. It will be seen that the supposition $p=0$, makes the given equations *Incomplete equations*.

Again, suppose $q=0$, p not being zero.

The first and third forms now become the same, as also the second and fourth.

(1) and (3). Here the first root becomes $-p+p$, or 0, and the second root $-p-p$, or $-2p$.

(2) and (4). Here the first root becomes $+p+p$, or $+2p$, and the second root $+p-p$, or 0.

Finally, suppose $p=0$ and $q=0$.

Under this hypothesis, both roots, in each form, reduce to 0.

The suppositions now made embracing all that can be made on the relative values of p and q , the discussion is

complete, and it will be seen that the same results are obtained by discussing the roots of the four forms, as by discussing the forms themselves.

SIMULTANEOUS EQUATIONS OF THE SECOND DEGREE.

392. As already shown, any equation containing more than one unknown quantity is indeterminate, and, to determine the values of any number of unknown quantities, we must have given as many simultaneous equations as there are unknown quantities. The rules for elimination have been deduced from principles common to all equations, no matter what the degree may be. The treatment of higher equations by these rules is not, of course, as simple as the treatment of equations of the first degree, but it may be stated that, in general, *Elimination* is always possible, whatever the degree of the simultaneous equations; the *Solution* of a resulting equation may or may not be possible. We proceed to the discussion of simultaneous equations of the second degree.

A. TWO EQUATIONS, INVOLVING BUT TWO UNKNOWN QUANTITIES.

393. General Case.—*In general, the solution of two simultaneous equations of the second degree requires the solution of an equation of the fourth degree.*

Every equation of the second degree containing but two unknown quantities is of the form

$$Ax^2 + Bxy + Cx + Dy^2 + Ey + G = 0; \quad (1)$$

for we may add all the terms involving x^2 , and denote the algebraic sum of their coefficients by A, we may add all the terms involving xy , and denote the algebraic sum of their coefficients by B, and so for the other terms. Further, the equation being only of the second degree, cannot contain any other terms than those here indicated. If any equation should not contain all of the terms here indicated, we should know that the coefficients of the missing terms are, in each case, zero.

To discuss the most general case of two simultaneous equations of the second degree, containing two unknown quantities, let us take two equations of the most general form, $Ax^2+Bxy+Cx+Dy^2+Ey+G=0$. (1)

$$A'x^2+B'xy+C'x+D'y^2+E'y+G'=0. \quad (2)$$

Though we may select any method of elimination, let us take the method by substitution, as the shortest. Transposing, we obtain from (1), $Ax^2+(By+C)x=-Dy^2-Ey-G$. Solving with respect to x ,

$$x = -\frac{By+C}{2A} \pm \sqrt{\left(\frac{By+C}{2A}\right)^2 - \frac{Dy^2+Ey+G}{A}}$$

If, now, we substitute this value of x in (2), we shall obtain an equation, containing y^2 , and also radicals under which is found y^2 . To free this equation from radicals, then, we must square an expression containing y^2 , *i. e.*, we must obtain an equation containing y^4 , which equation will be of the fourth degree.

As we have not yet learned how to solve equations of the fourth degree, we shall not proceed further with the discussion of the general case. SPECIAL CASES, however, admit of solution by the use only of principles already proved, and to the discussion of these we proceed at once.

I. *If two equations, one of the first degree, and the other of the second, contain two unknown quantities, and no more, the values of these quantities may be determined.*

Let $Ax^2+Bxy+Cx+Dy^2+Ey+G=0$ (1) represent any equation of the second degree, and $Mx+Ny+L=0$ (2) represent any equation of the first degree, each equation containing two unknown quantities, and no more.

The value of x , from (2), is $-\frac{Ny+L}{M}$ (3). If this value

be substituted for x in (1), the highest power of y in the resulting equation will be the second, *i. e.*, we shall have an equation of the second degree involving only y . Solving this equation, we deduce the value of y , and this value, substituted in (3), gives the value of x .

EXAMPLES.

1. Given, $x+2y=11$ (1), and $3x^2-2y^2+5=0$ (2); required, the values of x and y .

From the first equation, we find $x=11-2y$; substituting this expression for x , in the second, we obtain

$$363-132y+12y^2-2y^2+5=0; y^2-\frac{132}{10}y=-\frac{368}{10}$$

$$y=\frac{66}{10}\pm\sqrt{-\frac{3680}{100}+\frac{4356}{100}}=\frac{676}{100}=\frac{66}{10}\pm\frac{26}{10}=\frac{92}{10}\text{ or }\frac{40}{10}$$

$$=9.2\text{ or }4$$

Hence, $x=11-18.4=-7.4$; $x=11-8=3$.

$$2. \begin{cases} 2x^2-3xy+5x+7y^2-2y-101=0 & (1) \\ 5x-y=11 & (2) \end{cases}$$

From (2), $y=-11+5x$. Substituting this value of y in (1), we obtain,

$$2x^2-15x^2+33x+5x+175x^2-770x+847-10x+22-101=0.$$

$$\text{Reducing, } 162x^2-742x=-768; x^2-\frac{742}{162}x=-\frac{768}{162};$$

$$x=\frac{371}{162}\pm\sqrt{-\frac{768}{162}+\left(\frac{371}{162}\right)^2}=\frac{-124416+137641}{(162)^2}=\frac{13225}{(162)^2}$$

$$=\frac{371}{162}\pm\frac{115}{162}=\frac{486}{162}\text{ or }\frac{256}{162}=3\text{ or }\frac{128}{81}.$$

Substituting this value of x in (2), we find $y=4$ or $-\frac{251}{81}$.

$$3. \begin{cases} x+2y=9 & y=4\text{ or }2 \\ x^2+2y^2=33 & x=1\text{ or }5 \end{cases} \quad 4. \begin{cases} x-3y=1 & x=4\text{ or }-5 \\ 2x^2-6y^2=26 & y=1\text{ or }-2 \end{cases}$$

$$5. \begin{cases} 2x-\frac{3(x-y)}{6}=4+x & x=2\text{ or }5. \\ y-1=\frac{x+3y}{x+2} & y=6\text{ or }3. \end{cases}$$

$$6. \begin{cases} x^2-\frac{5}{2}y^2=10-\frac{xy}{2} & x=5\text{ or }-\frac{37}{4}. \\ \frac{2}{3}x=\frac{1+3y}{3} & y=3\text{ or }-\frac{13}{2}. \end{cases}$$

$$7. \begin{cases} \frac{10x+y}{8} = xy & x=2 \text{ or } -\frac{1}{3} \\ y-x=2 & y=4 \text{ or } \frac{5}{3} \end{cases} \quad 8. \begin{cases} \frac{x}{m} + \frac{y}{n} = 1. & (1) \\ \frac{m}{x} + \frac{n}{y} = 4. & (2) \end{cases}$$

From (1) $\frac{y}{n} = 1 - \frac{x}{m} = \frac{m-x}{m}$. Substituting in (2), remembering that $\frac{n}{y}$ is the reciprocal of $\frac{y}{n}$, we obtain $\frac{m}{x} + \frac{m}{m-x} = 4$;
or, $m^2 = 4mx - 4x^2$; whence, $x^2 - mx = -\frac{m^2}{4}$

$$x = \frac{m}{2} \pm \sqrt{-\frac{m^2}{4} + \frac{m^2}{4}} = 0.$$

$$x = \frac{m}{2}. \text{ Substituting, } y = \frac{n}{2}.$$

$$9. \begin{cases} 3x+2y=12 & x=-23 \text{ or } 2. \\ 8x^2-2y^2+xy=20. & y=40\frac{1}{2} \text{ or } 3. \end{cases}$$

$$10. \begin{cases} 6x+7y+xy=38 & x=1 \text{ or } 3. \\ 2x+y=10-y. & y=4 \text{ or } 2. \end{cases}$$

$$11. \begin{cases} \frac{3x-y}{5} = .06x + .025y & x=0 \text{ or } -1. \\ x-y = \frac{.5y+2.25xy}{3} & y=0 \text{ or } -2.4. \end{cases}$$

DEFINITION.—An equation is homogeneous when the sum of the exponents of the unknown quantities is the same in every term into which these quantities enter.

II.—From two homogeneous equations of the second degree we may always deduce a single equation of the second degree, by substituting for one of the unknown quantities a new unknown quantity multiplied into the other.

Let $Ax^2 + Bxy + Cy^2 = G$ (1) and

$A'x^2 + B'xy + C'y^2 = G'$ (2) be any two homogeneous equations of the second degree. Placing $y=px$, and substituting in (1) and (2), we obtain

$$Ax^2 + Bpx^2 + C p^2 x^2 = G \quad (3)$$

$$A'x^2 + B'px^2 + C'p^2 x^2 = G' \quad (4)$$

From (3) we have, $(A+Bp+Cp^2)x^2=G$, or $x^2=\frac{G}{A+Bp+Cp^2}$.

$$\text{And from (4), } x^2=\frac{G'}{A'+B'p+C'p^2}$$

Hence, since the equations are simultaneous,

$$\frac{G}{A+Bp+Cp^2}=\frac{G'}{A'+B'p+C'p^2}$$

$$\text{or } (A+Bp+Cp^2)G'=(A'+B'p+C'p^2)G \quad (5),$$

(5) being a complete equation, in which p is the unknown quantity, we may solve it, and, finding p , we may substitute its value in (3), and, solving the resulting incomplete equation, obtain x . Substituting the value of x in (1), we find the value of y from the resulting complete equation of the second degree.

EXAMPLES.

1. $2x^2+3xy+y^2=20$ (a) Substituting px for y , in both
 $5x^2+4y^2=41$ (b) equations, we obtain

$$2x^2+3px^2+p^2x^2=20 \quad (c) \qquad 5x^2+4p^2x^2=41 \quad (d)$$

$$\text{From (c) } x^2=\frac{20}{2+3p+p^2}; \qquad \text{From (d) } x^2=\frac{41}{5+4p^2}$$

Hence, the equations being simultaneous,

$$\frac{20}{2+3p+p^2}=\frac{41}{5+4p^2}; \text{ or } 100+80p^2=82+123p+41p^2$$

$$\text{Reducing, } 39p^2-123p=-18;$$

$$p^2-\frac{123}{39}p=-\frac{18}{39}; \qquad \text{or } p^2-\frac{41}{13}p=-\frac{6}{13} \quad (e)$$

Solving (e),

$$p=\frac{41}{26} \pm \sqrt{-\frac{6}{13}+\frac{1681}{(26)^2}}=\frac{-312+1681}{(26)^2}=\frac{1369}{(26)^2}$$

$$=\frac{41}{26} \pm \frac{37}{26}=\frac{78}{26} \text{ or } \frac{4}{26}=3 \text{ or } \frac{2}{13}$$

Substituting in (d), we obtain:

$$\begin{array}{rcl}
 5x^2 + 36x^2 = 41 & \text{and} & 5x^2 + \frac{16}{169}x^2 = 41 \\
 41x^2 = 41 & & 861x^2 = 6929 \\
 x^2 = 1 & & 21x^2 = 169 \\
 x = \pm 1 & & x^2 = \frac{169}{21} \\
 & & x = \pm \frac{13}{\sqrt{21}}
 \end{array}$$

Since $y = px$

$$y = \pm 1 \times 3 = \pm 3 \text{ and } y = \pm \frac{13}{\sqrt{21}} \times \frac{2}{13} = \pm \frac{2}{\sqrt{21}}.$$

The value of y might also be obtained by substituting the value of x in (b) or (a). Thus, taking the first value of x , $5 + 4y^2 = 41$; or $4y^2 = 36$. Hence, $y^2 = 9$ and $y = \pm 3$.

Taking the second value, $\frac{845}{21} + 4y^2 = 41$;

$$\text{or } y^2 = \frac{4}{21} \text{ and } y = \pm \frac{2}{\sqrt{21}}$$

2. $\left. \begin{array}{l} 3x^2 - 2xy + y^2 = 6 \\ 7x^2 - 11xy + 5y^2 = 19 \end{array} \right\}$ Comparing with equations (1) and (2) of the discussion, we see that in this $A=3$, $B=-2$, $C=1$, and $G=6$

$$A'=7, B'=-11, C'=5, \text{ and } G'=19$$

Hence (5) becomes $57 - 38p + 19p^2 = 42 - 66p + 30p^2$.

Reducing, we obtain $11p^2 - 28p = 15$, or $p^2 - \frac{28}{11}p = \frac{15}{11}$.

$$\text{Hence } p = \frac{14}{11} \pm \sqrt{\frac{15}{11} + \frac{196}{121}} = \frac{361}{121} = \frac{14}{11} \pm \frac{19}{11} = 3 \text{ or } -\frac{5}{11}.$$

Substituting the value 3 in the expression for x^2 , we obtain

$$x^2 = \frac{6}{3-6+9} = 1 \therefore x = \pm 1.$$

Taking the second value of p ,

$$x^2 = \frac{6}{3 + \frac{10}{11} + \frac{25}{121}} = \frac{6}{\frac{363 + 110 + 25}{121}} = \frac{121}{83} \therefore x = \pm \frac{11}{\sqrt{83}}.$$

Since $y=px$, $y=\pm 1 \times 3=\pm 3$; $y=\pm \frac{11}{\sqrt{83}} \times -\frac{5}{11}=\mp \frac{5}{\sqrt{83}}$.

$$3. \begin{cases} \frac{x^2+xy}{4}=3 & x=\pm 3 \text{ or } \pm \frac{8}{\sqrt{6}} \\ \frac{xy}{2}=\frac{1+2y^2}{2} & y=\pm 1 \text{ or } \pm \frac{1}{\sqrt{6}} \end{cases}$$

$$4. \begin{cases} x^2+xy-y^2=5 & x=2 \text{ or } -2. \\ x^2-\frac{2(xy+y^2)}{3}=2 & y=1 \text{ or } -1. \end{cases}$$

$$5. \begin{cases} 4x^2-3xy+y^2=2 \\ x^2-y^2=-3 \end{cases} \quad 6. \begin{cases} x^2-\frac{5xy}{7}+y^2=45. \\ x^2-25y^2=24. \end{cases}$$

$$7. \begin{cases} (13x)^2+2y^2=177 \\ (2y)^2-13x^2=3 \end{cases} \quad \begin{matrix} x=\pm 1. \\ y=\pm 2. \end{matrix}$$

$$8. \begin{cases} x^2-xy=25+\frac{xy}{2} & x=10 \text{ or } 5\sqrt{-2}. \\ 3x^2-(20+6y^2)=130 & y=5 \text{ or } 5\sqrt{-2}. \end{cases}$$

$$9. \begin{cases} x^2+3w^2=28 \\ wx-x^2-\frac{1}{2}=17-\frac{3x^2}{2} \end{cases} \quad \begin{matrix} x=5 \text{ or } \sqrt{21}. \\ w=1 \text{ or } \frac{\sqrt{21}}{3}. \end{matrix}$$

$$10. \left\{ \begin{aligned} \frac{x^2}{2} + \frac{5xy}{3} - \frac{y^2}{2} &= \frac{5}{2} \\ 5x^2 - 10y^2 + xy &= 11 \end{aligned} \right\} \text{ Verify.}$$

SYMMETRICAL EQUATIONS.

Symmetrical Equations are those in which the unknown quantities are similarly involved; i. e., the unknown quantities may interchange positions without altering the expressions.

ILLUSTRATION. $x^2+y^2+z^2=36$, $x^2+xy+yz+y^2+z^2=20-yz$, $x+y=7$, $x^2+y^2=10$, and $x^4+x^2y^2+y^4=56$, are symmetrical. $x-y=8$ is not a symmetrical equation, if x and y be considered the unknown quantities; for, though there is a $+x$ in the equation, there is no $+y$, and $-y$ occurs, but $-x$ does not.

III. *Two symmetrical equations of the second degree containing but two unknown quantities may be solved by substituting for one of the quantities the sum of two auxiliary unknown quan-*

tities, and for the other the difference of the same auxiliaries; we can then deduce from the equations a single, determinate equation of the second degree.

Let $ax^2 + ay^2 + Bxy + cx + cy + G = 0$ (1)

$a'x^2 + a'y^2 + B'xy + c'x + c'y + G' = 0$ (2) represent any symmetrical equations of the second degree, containing only two unknown quantities. Factoring, and writing A for a, C for c, and A' and C' for a' and c', respectively, we have

$$A(x^2 + y^2) + Bxy + C(x + y) + G = 0 \quad (3)$$

$$A'(x^2 + y^2) + B'xy + C'(x + y) + G' = 0 \quad (4)$$

Now the sum of x and y being $2s$, and their difference $2z$, place $x = s + z$ and $y = s - z$, s and z being auxiliary unknown quantities; we then have, since

$$x^2 = s^2 + 2sz + z^2; y^2 = s^2 - 2sz + z^2; xy = s^2 - z^2, \text{ etc.}$$

$$A(2s^2 + 2z^2) + B(s^2 - z^2) + C \cdot 2s + G = 0 \quad (5)$$

$$A'(2s^2 + 2z^2) + B'(s^2 - z^2) + C' \cdot 2s + G' = 0 \quad (6) \text{ or,}$$

$$(2A + B)s^2 + (2A - B)z^2 + 2Cs + G = 0 \quad (7)$$

$$(2A' + B')s^2 + (2A' - B')z^2 + 2C's + G' = 0 \quad (8)$$

Denoting the coefficients of s^2 and z^2 , in (7), by M and N, and in (8), by M' and N', $M s^2 + N z^2 + 2Cs + G = 0$ (9)

$$M' s^2 + N' z^2 + 2C's + G' = 0 \quad (10)$$

Eliminating z^2 by subtraction (365) from (9) and (10), we have

$$(MN' - M'N)s^2 + 2(CN' - C'N)s + GN' - G'N = 0 \quad (11)$$

which is an ordinary complete equation of the second degree, and may therefore be solved as in (383). Having s , by substituting in (5) or (6), we obtain the value of z , whence, s and z being known, x and y are known.

EXAMPLES.

$$x^2 + y^2 = 52 \quad (1); x + y + xy = 34 \quad (2).$$

Place $\begin{matrix} x = s + z \\ y = s - z \end{matrix}$. Then $x^2 + y^2 = 2s^2 + 2z^2 = 52$, or $s^2 + z^2 = 26$ (3)

$$x + y + xy = 2s + s^2 - z^2 = 34 \quad (4). \text{ Eliminating } z^2, (365)$$

$$2s + 2s^2 = 60, s^2 + s = 30 \therefore s = 5 \text{ or } -6.$$

$$25 + z^2 = 26, \text{ whence } z = \pm 1$$

Substituting in (3), $36 + z^2 = 26$, whence $z = \pm \sqrt{-10}$.

Hence $x=6$ or 4 , and $-6 \pm \sqrt{-10}$
 $y=4$ or 6 , and $-6 \mp \sqrt{-10}$.

$$\begin{array}{l} (1.) \\ \left\{ \begin{array}{l} 5x^2+5y^2+4xy=356 \\ x^2+y^2+x+y=62. \end{array} \right. \end{array} \quad \begin{array}{l} x=4 \text{ or } 6. \\ y=6 \text{ or } 4. \end{array} \quad \begin{array}{l} (2.) \\ \left\{ \begin{array}{l} 7x^2+7y^2+xy=37 \\ -x^2-y^2+x+y=-2. \end{array} \right. \end{array}$$

$$\begin{array}{l} (3.) \\ \left\{ \begin{array}{l} \frac{1}{x} + \frac{1}{y} = \frac{1}{5} \\ \frac{10}{xy} = \frac{1}{18} \end{array} \right. \end{array} \quad \begin{array}{l} x=6 \text{ or } 30 \\ y=30 \text{ or } 6 \end{array} \quad \begin{array}{l} (4.) \\ \left\{ \begin{array}{l} x^2+y^2-(x+y)=78 \\ xy+x+y=39 \end{array} \right. \end{array} \quad \begin{array}{l} x=9 \text{ or } 3 \\ y=3 \text{ or } 9 \end{array}$$

5. $\begin{cases} 8(x^2+y^2)-3xy=81+x+y & (1). \\ x^2+y^2-(x+y)=8 & (2). \end{cases}$

Let $x=z+s$; whence, $x^2=z^2+2zs+s^2$
 $y=z-s \quad y^2=z^2-2zs+s^2$.

Substituting for x and y in (1) and (2) their values, $z+s$ and $z-s$, respectively, we obtain

$$16z^2+16s^2-3z^2+3s^2=81+2z; \text{ or, } 13z^2+19s^2-2z=81 \quad (3)$$

$$2s^2+2s^2-2z=8; \text{ or, } z^2+s^2-z=4 \quad (4)$$

Multiplying both members of (4) by 19, and subtracting the results from the members of (3), we obtain

$$-6z^2+17z=5 \text{ or } 6z^2-17z=-5$$

Reducing, $z^2 - \frac{17}{6}z = -\frac{5}{6}$

$$\begin{aligned} z &= \frac{17}{12} \pm \sqrt{-\frac{5}{6} + \frac{289}{144}} = \frac{-120+289}{144} \\ &= \frac{17}{12} \pm \frac{13}{12}, = \frac{30}{12} \text{ or } \frac{4}{12}, = \frac{5}{2} \text{ or } \frac{1}{3}. \end{aligned}$$

Substituting the first value of z in (4), we obtain

$$\frac{25}{4} - \frac{10}{4} - \frac{16}{4} = -s^2; \text{ or } -\frac{1}{4} = -s^2 \therefore s^2 = \frac{1}{4} \text{ and } s = \pm \frac{1}{2}$$

Hence, $x=z+s = \pm \frac{1}{2} + \frac{5}{2} = \frac{6}{2} \text{ or } \frac{4}{2}, = 3 \text{ or } 2$.

$$y=z-s = \frac{5}{2} \mp \frac{1}{2} = \frac{4}{2} \text{ or } \frac{6}{2}, = 2 \text{ or } 3.$$

Taking the second value of z , we have

$$\frac{1}{9} + s^2 - \frac{1}{3} = 4; \text{ or } 9s^2 = 36 + 3 - 1 = 38. \therefore s^2 = \frac{38}{9} \text{ and } s = \pm \sqrt{\frac{38}{9}}$$

$$\text{Hence } x = \frac{1}{3} \pm \sqrt{\frac{38}{9}} \quad y = \frac{1}{3} \mp \sqrt{\frac{38}{9}}$$

$$6. \begin{cases} 4x + 4y = 3xy & x=4 \text{ or } 2, \text{ and } -\frac{13}{6} \pm \frac{\sqrt{377}}{6}. \\ x + y + x^2 + y^2 = 26 & y=2 \text{ or } 4, \text{ and } -\frac{13}{6} \mp \frac{\sqrt{377}}{6}. \end{cases}$$

$$7. \begin{cases} x + y + xy = 11. & x=2 \text{ or } 3. \\ x^2 + y^2 + x + y = 18. & y=3 \text{ or } 2. \end{cases}$$

$$8. \begin{cases} x^2 - x + y^2 - y = 32. & x=5 \text{ or } 4. \\ 4xy + x^2 + y^2 = 121. & y=4 \text{ or } 5. \end{cases}$$

$$9. \begin{cases} xy = a. & x = \frac{\pm \sqrt{b+2a} \pm \sqrt{b-2a}}{2}. \\ x^2 + y^2 = b. & y = \frac{\pm \sqrt{b+2a} \mp \sqrt{b-2a}}{2}. \end{cases}$$

10. $\begin{cases} x^2 + xy + y^2 = a & (1) \\ x^2 + y^2 = b & (2) \end{cases}$ Subtracting (2) from (1), we find $xy = a - b$, and from this equation and (2), we obtain the values of x and y , as in example 9.

IV. When we have two equations, involving different powers of the same algebraic expression, or equations from which we may deduce an equation involving the powers of any algebraic expression, we may often solve by regarding this expression as the unknown quantity.

$$\text{Thus, if we have } \begin{cases} x^2 + y^2 + 2xy + 2x = 120 - 2y & (1) \\ xy - y^2 = 8 & (2) \end{cases}$$

we may write (1), $(x+y)^2 + 2(x+y) = 120$ (3).

Solving (3) with regard to $(x+y)$, i. e., regarding $x+y$ as the unknown quantity, we find:

$$x+y = -1 \pm \sqrt{121} = 10 \text{ or } -12; \quad \begin{matrix} x+y = 10 & (4) \\ x+y = -12 & (5) \end{matrix}$$

Combining (4) and (2), by I, we obtain

$$\begin{matrix} 10y - 2y^2 = 8 & -12y - 2y^2 = 8 \\ y^2 - 5y = -4 & y^2 + 6y = -4 \end{matrix}$$

$$\text{whence, } \begin{matrix} y = 4 \text{ or } 1 & y = -3 \pm \sqrt{5}. \end{matrix}$$

Substituting these values of y , we find

$$x = 6 \text{ or } 9, \text{ or } -9 \mp \sqrt{5}.$$

EXAMPLES.

$$1. \begin{cases} x^2 + y^2 - x + y = 18 \\ xy = 9 \end{cases} \quad 2. \begin{cases} x^2 - 2xy + y^2 - 7x = 15 - 7y \\ 8xy - 2y^2 = 10 \end{cases}$$

3. $\begin{cases} x^2 + x + y = 18 - y^2 \\ xy = 6 \end{cases}$ (1) Multiplying both members of
(2) (2) by 2, and adding to (1),
we obtain $x^2 + 2xy + y^2 + x + y = 30$

$$(x+y)^2 + (x+y) = 30 \quad \therefore x+y = -\frac{1}{2} \pm \sqrt{\frac{121}{4}} \\ = 5 \text{ or } -6$$

We may now solve by (I): $x=3$ or 2 ; or $-3 \pm \sqrt{3}$
 $y=2$ or 3 ; or $-3 \mp \sqrt{3}$

$$4. \begin{cases} y^2 - x^2 + xy = 11 \\ 2x^2 + xy = 14 \end{cases}$$

$$5. \begin{cases} (x+y)^2 - 2x^2 = 49 \\ 3x^2 + 4(x+y)^2 = 372 \end{cases} \quad \begin{matrix} x = \pm 4 \\ y = \pm 5 \text{ or } \pm 13 \end{matrix}$$

$$6. \begin{cases} -x^2 + 6xy - 9y^2 + 4x - 12y = 4 \\ x^2 - 2xy + 3y^2 - 4x + 5y = 53 \end{cases} \quad \begin{matrix} x = 11 \text{ or } -\frac{15}{2} \\ y = 3 \text{ or } -\frac{19}{6} \end{matrix}$$

$$7. \begin{cases} x^2 - 2xy + y^2 = x - y^2 \\ x^2 = 5 + y^2 \end{cases}$$

$$8. \begin{cases} 4(x+5)^2 - 14 = 2xy - 18x + 10y - 92 \\ 10y + 2xy - 18x - 90 = -2 + y^2 - 18y + 81 \end{cases} \quad \begin{matrix} (1). \\ (2). \end{matrix}$$

(1) May be written: $4(x+5)^2 - 12 = 2(x+5)(y-9)$ (4).

(2) May be written: $-2(x+5)(y-9) = 2 - (y-9)^2$ (5).

Regarding $2(x+5)$ and $y-9$ as the unknown quantities,
we may solve (4) and (5) by II, since they are homogeneous.

We thus obtain,

$$2(x+5) = \pm \frac{6\sqrt{2}}{\sqrt{7}}; \quad x+5 = \pm 3\sqrt{\frac{2}{7}}; \quad \text{and } y-9 = \pm \sqrt{\frac{2}{7}}.$$

$$\text{Whence } x = -5 \pm 3\sqrt{\frac{2}{7}}; \quad \text{and } y = 9 \mp \sqrt{\frac{2}{7}}.$$

9.

$$\begin{cases} 9(x^2 - 2bx + b^2) + 16y^2 + 32dy + 16d^2 - 3(x-b) = 12 + 4y + 4d \\ 12(xy - bd - by + dx) = 9 \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Regarding $3(x-b)$ and $4(y+d)$ as the unknown quantities, (1) and (2) are symmetrical, and may, therefore, be solved by III. Multiplying (2) by 2, and adding, member to member, to (1), and solving the resulting equation with regard to $3(x-b)+4(y+d)$, we obtain $3(x-b)+4(y+d)=6$ or -5 .

Taking the positive value, we find $3(x-b)=4(y+d)=\pm 3$, or $x-b=\pm 1$, $x=b\pm 1$, and $y+d=\pm \frac{3}{4}$, $y=-d\pm \frac{3}{4}$.

Taking the negative value, we obtain,

$$\begin{aligned} 6(x-b) &= -5 \pm \sqrt{-11} & 8(y+d) &= -5 \mp \sqrt{-11} \\ x-b &= \frac{-5 \pm \sqrt{-11}}{6} & y+d &= \frac{-5 \mp \sqrt{-11}}{8} \\ x &= \frac{6b-5 \pm \sqrt{-11}}{6} & y &= -\frac{8d+5 \pm \sqrt{-11}}{8} \end{aligned}$$

$$10. \begin{cases} 4x^2-14x+21y-132=-9y^2 & (1). \\ xy=6 & (2). \end{cases}$$

(1) May be written: $4x^2-7(2x-3y)+9y^2=132$ (3).

Multiplying both members of (2) by 12, $12xy=72$ (4).

Subtracting (4) from (3), member from member, we obtain $4x^2-12xy+9y^2-7(2x-3y)=60$ (5). Solving (5) with regard to $2x-3y$, we obtain,

$$2x-3y = \frac{7}{2} \pm \sqrt{60 + \frac{49}{4}} = \frac{7}{2} \pm \frac{17}{2} = 12 \text{ or } -5.$$

Taking $2x-3y=12$ (6), and combining with (2), we find $y^2+4y=4$; whence, $y=-2 \pm \sqrt{8} = -2(1 \pm \sqrt{2})$.

$$\begin{aligned} \text{Substituting in (6), } 2x+6(1 \mp \sqrt{2}) &= 12; & 2x &= 12-6 \pm 6\sqrt{2} \\ & & &= 6(1 \pm \sqrt{2}) \\ x &= 3(1 \pm \sqrt{2}) \end{aligned}$$

For $2x-3y=-5$ (7), we have, after combining with (2), $3y^2-5y=12$; Substituting in (7),

$$\begin{aligned} y^2 - \frac{5}{3}y &= 4 & 2x-9 &= -5 & \text{and } 2x+4 &= -5 \\ y &= \frac{5}{6} \pm \sqrt{\frac{169}{36}} = 3, \text{ or } -\frac{4}{3}. & 2x &= 4 & 2x &= -9 \\ & & x &= 2 & x &= -\frac{9}{2} = -4\frac{1}{2} \end{aligned}$$

$$11. \begin{cases} x^2 - 3xy - y^2 = 31 - x + 2y \\ xy - 5y^2 = 1 \end{cases} \quad \text{Verify.}$$

V. Special Cases.

The solution of simultaneous equations of the second degree is possible in some cases, not yet discussed. Of these cases it is, however, neither convenient nor important to attempt a classification. We shall illustrate the most common artifices, leaving the pupil to depend on his own skill in applying these, or inventing new devices for the particular examples he may meet. It often happens that we may, by eliminating (365) one of the terms, deduce an equation that we may solve directly. Thus, having $x^2 + 3y^2 = 40$ we deduce at once $7x^2 = 49 \therefore x^2 = 7$ and $x = \pm\sqrt{7}$. We should in general try the common methods of eliminating and solving, before passing to special expedients.

EXAMPLES.

$$1. \begin{cases} x+y+x^2+y^2=a & (1.) \\ x-y+x^2-y^2=b & (2.) \end{cases} \quad \text{Adding (1) and (2),}$$

$$x^2 + x = \frac{a+b}{2}; \quad x = \frac{-1 \pm \sqrt{1+2a+2b}}{2}; \quad \text{subtracting (2) from (1),}$$

$$\text{member from member, } y^2 + y = \frac{a-b}{2}; \quad y = \frac{-1 \pm \sqrt{1+2a-2b}}{2}.$$

$$2. \begin{cases} \sqrt{\frac{3x}{x+y}} + \sqrt{\frac{x+y}{3x}} = 2 & (1.) \\ xy - x - y = 54 & (2.) \end{cases} \quad \text{Squaring, we obtain, from (1),}$$

$$\frac{3x}{x+y} + \frac{x+y}{3x} + 2 = 4 \text{ or } 9x^2 + (x+y)^2 - 6x(x+y) = 0,$$

$$[3x - (x+y)]^2 = 0, \quad 3x - (x+y) = 0 \therefore 3x = x+y$$

$$2x = y;$$

$$\text{Substituting, in (2), } 2x^2 - 3x = 54$$

$$y = 2x.$$

$$x^2 - \frac{3}{2}x = 27$$

$$= 12 \text{ or } -9.$$

$$x = \frac{3}{4} \pm \sqrt{\frac{441}{16}} = \frac{3}{4} \pm \frac{21}{4} = 6 \text{ or } -\frac{18}{4} = -\frac{9}{2}.$$

$$3. \begin{cases} x^2+1=ax & (1) \\ x^2-1=xy & (2) \end{cases} \quad \text{Write (1): } x+\frac{1}{x}=a \quad (3)$$

$$\text{Write (2): } x-\frac{1}{x}=y \quad (4)$$

$$\text{Adding,} \quad \frac{2x}{2}=a+y \quad (5)$$

$$\text{Subtracting,} \quad \frac{2}{x}=a-y \quad (6)$$

$$\text{Multiplying (5) and (6),} \quad 4=a^2-y^2 \therefore y=\pm\sqrt{a^2-4}.$$

$$4. \begin{cases} \frac{xy}{\sqrt{x}}=48 & (1.) \\ \sqrt{\frac{x}{y}} & (2.) \end{cases} \quad \text{Divide (1) by (2).}$$

$$5. \begin{cases} 10x^2+3xy-18y^2+25x-2y=-95 & (1) \\ 6x^2+2xy-18y^2+26x+28y=28-4x^2-xy & (2) \end{cases}$$

$$\text{Write (1): } (2x+3y)(5x-6y)+25x-2y=-95 \quad (3)$$

Transpose in (2), and write thus:

$$(2x+3y)(5x-6y)+26x+28y=28 \quad (4)$$

$$\text{Subtracting (3) from (4), member from member,} \\ x+30y=123 \quad (5)$$

which may be combined (I) with either (1) or (2).

$$7. \begin{cases} x+\sqrt{xy}+y=22 & (1) \\ x^2+xy+y^2=132 & (2) \end{cases} \quad \text{These equations may be solved}$$

by transforming (1) thus; $\sqrt{xy}=22-(x+y)$, squaring, and combining the resulting equation with (2), by (III) or (IV), but it is better to divide (2) by (1), member by member, thus obtaining $x-\sqrt{xy}+y=6$ (3), which, added to (1), gives $x+y=14$ (4). Combining (4) with (1), we have

$$\sqrt{xy}+14=22 \therefore \sqrt{xy}=8 \therefore x=\frac{64}{y};$$

substituting in (4), $y^2-14y=-64$.

$$8. \begin{cases} x^2+xy=a & (1) \\ y^2+xy=b & (2) \end{cases}$$

Adding, $(x+y)^2=a+b \therefore x+y=\pm\sqrt{a+b}$ (3). Subtracting, $x^2-y^2=a-b \therefore (x+y)(x-y)=a-b$ (4). Dividing (4) by (3), member by member, $x-y=\frac{a-b}{\pm\sqrt{a+b}}$ (5). Adding (5)

to (3), member to member, $2x = \frac{2a}{\pm\sqrt{a+b}} \therefore x = \pm \frac{a}{\sqrt{a+b}}$.

$$\text{Hence } y = \pm \frac{b}{\sqrt{a+b}}.$$

The equations may also be solved by **III**, **II**, and, after the first step, by **I**.

$$9. \begin{cases} x^2 + y^2 = \frac{5}{2}xy. & x=0 \text{ or } 4 \text{ or } -2. \\ x-y = \frac{xy}{4}. & y=0 \text{ or } 2 \text{ or } -4. \end{cases}$$

Regarding $-y$ as an unknown quantity, the equations are symmetrical.

$$10. \sqrt{\frac{12x}{4x+2y}} = \frac{10}{27} - \sqrt{\frac{6x}{2x+y}}; \quad 3xy - \frac{6x+3y}{4} = 30.$$

B. THREE OR MORE EQUATIONS INVOLVING THREE OR MORE UNKNOWN QUANTITIES.

394. The solution of the general case is, of course, impossible, as equations of the fourth degree would result from the first steps of the elimination.

Special cases admit of solution; we shall call attention to some of these, and illustrate some by examples, our limits forbidding an exhaustive treatment.

If we have several equations of the first degree, and one of the second, we can always solve, for the problem is finally reduced to the case **A I**, after all the quantities except two have been eliminated from the equations of the first degree.

In general, if we have m equations of the first degree and n of the second, involving in all $m+n$ unknown quantities, we may, (**370**) from the equations of the first degree, deduce a single equation, containing not more than $n+1$ unknown quantities. This we can combine with one of the equations of the second degree, obtaining a single equation of the second degree, having eliminated an unknown quantity. We now have n equations of the second degree. If

$n=1$, or 2 , we may solve by methods already discussed. If $n>2$, and we meet at any step of the process with equations to which **A I—V** will not apply, we must depend for the solution entirely on such special expedients as may be suggested by the nature of the case.

EXAMPLES.

$$1. \begin{cases} 3x-2y+z=4 & (1) \\ 5x+3y-4z=3 & (2) \\ x^2-yz+z^2=8 & (3) \end{cases} \quad \text{From (1) and (2)} \quad \begin{cases} 15x-10y+5z=20 \\ 15x+9y-12z=9 \\ \hline -19y+17z=11 \\ y=\frac{17z-11}{19} \end{cases}$$

Substituting in (3), $x^2 - \frac{17z^2-11z}{19} + z^2 = 8$ (4).

From (1) and (2), eliminating y , we obtain $19x-5z=18$;
whence, $x = \frac{18+5z}{19}$, which, substituted in (4), gives

$$\frac{324+180z+25z^2}{(19)^2} - \frac{17z^2-11z}{19} + z^2 = 8;$$

$$\text{or } 324+180z+25z^2-323z^2+209z+361z^2=2888;$$

$$\text{or } 63z^2+389z=2564.$$

Solving this equation, we may obtain z , which, substituted in the value of x , will give x , and this, substituted in (1), will give the value of y .

$$2. \begin{cases} x+y+z=6 \\ z+4x+2y=2z+5 \\ x^2+y^2+z^2=14 \end{cases} \quad \begin{matrix} x=1 \\ y=2 \\ z=3. \end{matrix}$$

$$3. \begin{cases} x^2=y^2+z^2-2nys & (1) \\ x+y+z=a & (2) \\ yz=b & (3) \end{cases} \quad \begin{matrix} \text{Adding } 2yz-2yz \text{ to the second} \\ \text{member of (1), we obtain} \\ x^2=(y+z)^2-2(1+n)yz & (4); \end{matrix}$$

whence, substituting the values of $y+z$ and yz from (2)

$$\text{and (3), } x^2=(a-x)^2-2(1+n)b. \therefore x = \frac{a^2-2(1+n)b}{2a}.$$

Adding $2yz-2yz$ to the second member of (1), we obtain also, $x^2=(y-z)^2+2(1-n)yz$; whence, x and yz being known, we may find $y-z$. Having, then, the sum and difference of y and z (from this operation and from (2)), we find y and z .

$$4. \begin{cases} x+y+z=6 & x=2 \text{ or } -18 \\ 3x=2yz & y=1 \text{ or } 3 \text{ and } 25.076 \text{ or } -1.076 \\ 2x^2+\frac{2}{3}yz=y^2+z^2 & z=3 \text{ or } 1 \text{ and } -1.076 \text{ or } 25.076. \end{cases}$$

$$5. \begin{cases} 3x^2-4z=6y+2 & x=2 \text{ or } -40 \\ 2z+x+2x^2=3+y & y=3 \text{ or } 1389 \\ 3y+3z+5=2x+x^2 & z=-2 \text{ or } -884. \end{cases}$$

$$6. \begin{cases} x^2+xy+xz=70 \\ y^2+xy+yz=20 \\ z^2+zx+zy=10 \end{cases} \quad \text{Verify.}$$

$$7. \begin{cases} xy=z & (1.) \\ yz=v & (2.) \\ xv=4 & (3.) \\ \frac{yv}{x}=27 & (4.) \end{cases} \quad \begin{array}{l} \text{Dividing (1) by (2), } \frac{x}{z}=\frac{z}{v} \therefore xv=z^2; \\ \text{Substituting in (3), we obtain} \\ z^2=4 \therefore z=\pm 2; \end{array}$$

Substituting this value of z in (1), $xy=\pm 2 \therefore x=\pm \frac{2}{y}$.
 Dividing (3) by (4), member by member, we obtain, $\frac{x^2}{y}=\frac{4}{27}$.
 or, substituting the value of x , $\frac{4}{y^3}=\frac{4}{27}$; whence, $\frac{1}{y^3}=\frac{1}{27}$;
 or $y^3=27 \therefore y=3$; hence we obtain, (2), $v=\pm 3$. $2=\pm 6$,
 (1) $x=\pm \frac{2}{3}$.

$$8. \begin{cases} u+z=2s & (1.) \\ x+y=2s' & (2.) \end{cases} \quad \begin{array}{l} u^2+x^2+y^2+z^2=4c^2 \quad (3.) \\ uz=xy \quad (4.) \end{array}$$

Adding $2uz$ to the first member of (3), and subtracting $2xy=2uz$, so as to preserve the equality of the members of (3), we obtain,

$$(5) u^2+2uz+z^2+x^2-2xy+y^2=4c^2, \text{ or } (u+z)^2+(x-y)^2=4c^2.$$

Substituting in (5) the value of $u+z$, from (1), we obtain

$$4s^2+(x-y)^2=4c^2; (x-y)^2=4c^2-4s^2 \therefore x-y=\pm 2\sqrt{c^2-s^2}.$$

From (2), $x+y=2s'$.

$$\text{Hence, } 2x=2s'\pm 2\sqrt{c^2-s^2}; x=s'\pm\sqrt{c^2-s^2}; y=s'\mp\sqrt{c^2-s^2}.$$

Similarly, by subtracting $2uz-2xy=0$, from the first member of (3), $(u-z)^2+(x+y)^2=4c^2$.

Since, (2), $x+y=2s$, $(u-z)^2=4c^2-4s'^2$; $u-z=\pm 2\sqrt{c^2-s'^2}$;
and, (1), $u+z=2s$,

hence, $2u=2s\pm 2\sqrt{c^2-s'^2}$; $u=s\pm \sqrt{c^2-s'^2}$; $z=s\mp \sqrt{c^2-s'^2}$.

$$9. \begin{cases} xyz=24. & xzv=40. & x=2. & z=4. \\ xyv=30. & yzv=60. & y=3. & v=5. \end{cases}$$

$$10. \begin{cases} 20y^2+4x^2-xz+5z^2=50 & (1). \\ 3x^2+xz+3z^2=25 & (2). \\ 9x^2-xz+9z^2-21y^2=50 & (3). \end{cases} \begin{array}{l} \text{Let } x=py \text{ and } z=qy. \\ \text{Then, from (1), (2), (3),} \\ \text{we obtain,} \end{array}$$

$$y^2(20+4p^2-pq+5q^2)=50; \text{ or } y^2=\frac{50}{20+4p^2-pq+5q^2} \quad (4);$$

$$y^2(3p^2+pq+3q^2)=25; \text{ or } y^2=\frac{25}{3p^2+pq+3q^2} \quad (5);$$

$$y^2(9p^2-pq+9q^2-21)=50; \text{ or } y^2=\frac{50}{9p^2-pq-9q^2-21} \quad (6);$$

$$\text{Hence, } \frac{50}{20+4p^2-pq+5q^2}=\frac{25}{3p^2+pq+3q^2} \quad (7);$$

$$\frac{50}{20+4p^2-pq+5q^2}=\frac{50}{9p^2-pq+9q^2-21} \quad (8);$$

$$\text{From (7), } 6p^2+2pq+6q^2=20+4p^2-pq+5q^2, \\ \text{or } 2p^2+3pq+q^2=20 \quad (9);$$

$$\text{From (8), } 20+4p^2-pq+5q^2=9p^2-pq+9q^2-21, \\ \text{or } 5p^2+4q^2=41 \quad (10).$$

Equations (9) and (10), being homogeneous, fall under (II).

Solving them, $p=\pm 1$ or $\pm \frac{13}{\sqrt{21}}$; $q=\pm 3$ or $\pm \frac{2}{\sqrt{21}}$.

By substituting these values in (4), (5), or (6), the values of y may be found, knowing which, we know x and z .

$$11. \begin{cases} xy=4x+4y & (1); \\ xz=5x+5z & (2); \\ yz=6y+6z & (3). \end{cases}$$

$$\text{Write (1): } \frac{1}{4}=\frac{1}{x}+\frac{1}{y}; (2): \frac{1}{5}=\frac{1}{x}+\frac{1}{z}; (3): \frac{1}{6}=\frac{1}{y}+\frac{1}{z}.$$

Eliminate by (365)

$$\frac{1}{y}-\frac{1}{z}=\frac{1}{20}; \frac{1}{y}+\frac{1}{z}=\frac{1}{6}; \frac{2}{y}=\frac{13}{60}; y=\frac{120}{13}.$$

SECTION III.

BINOMIAL EQUATIONS.

395. A Binomial Equation is one that contains only one power of an unknown quantity, exclusive of the zero power.

Such an equation can always be reduced to the form $x^n=q$, by uniting all the terms containing the unknown quantity into one term, and uniting all known terms into one term in the other member. Transposing, we have $x^n-q=0$, in which, the second member being zero, the first is a *binomial*, whence the name.

396. To Deduce a Rule for solving a Binomial Equation.

Let $x^n=q$ represent any binomial equation, reduced to its simplest form. We may have the cases: n positive, and entire or fractional; and n negative, and entire or fractional.

I. n positive and entire. It is obvious that, in this case, we may at once extract the n th root of the two members, obtaining $x=\pm\sqrt[n]{q}$ if n is even, and $x=\sqrt[n]{q}$ if n is odd.

II. n positive and fractional. Let $n=\frac{r}{s}$. Then $x^{\frac{r}{s}}=q$. Raising both members to the s th power, $x^r=q^s$; extracting the r th root of both members of this equation, $x=\sqrt[r]{q^s}$.

III. n negative, and either entire or fractional. Let $n=-p$. Then $x^{-p}=q$. Multiplying both members by x^p , we have $x^p=qx^p \therefore x^p=\frac{1}{q} \therefore x=\pm\sqrt[p]{\frac{1}{q}}$, or $\sqrt[p]{\frac{1}{q}}$, according as p is even or odd.

Hence we deduce the following

RULE.—Reduce the equation to the form $x^n=q$. The value of x is then $\sqrt[n]{q}$, with an exponent equal to $\frac{1}{n}$.

397. Corollary.—Two roots, equal with contrary signs, will be obtained if the number of units, in the exponent of x , is even; one root, having the sign of q , will be obtained when the number of these units is odd. If n is even and q negative, both roots obtained will be imaginary.

EXAMPLES.

$$1. 9x^6 - 3 + \frac{27x^3 - 18}{6} = \frac{9}{6} + 3x^3 + 3.$$

Reducing to the form $x^n = q$, we find $63x^3 = 63$, whence, $x^3 = 1 \therefore x = 1$.

$$2. 7x^{\frac{2}{3}} - 21 + \frac{5x^{\frac{2}{3}} - 4}{4} + 6 = 4x^{\frac{2}{3}} + 1. \quad x = \pm 8.$$

$$3. 5x^{-3} - 36 + \frac{12x^{-3} - 3}{7} = 3x^{-3} - \frac{2x^{-3} - 3}{5}. \quad x = \pm \frac{1}{3}.$$

$$4. 9y^{-\frac{3}{7}} - \frac{1}{8} - \frac{5y^{-\frac{3}{7}} + 3}{29} = 8y^{-\frac{3}{7}} + \frac{10y^{-\frac{3}{7}} - 3}{14}. \quad y = 128.$$

$$5. 27x^{-\frac{3}{5}} + 6 + \frac{5x^{-\frac{3}{5}}}{3} = \frac{3}{27} + \frac{2x^{-\frac{3}{5}}}{3} + 81x^{-\frac{3}{5}}. \quad x = \pm 243.$$

398. If an equation involves but one power of any algebraic expression whatever, we may, of course, find the value of this expression from (396), and may thus often find one or more roots of an equation very high in degree, involving different powers of the unknown quantity. To take a single and easy example, let

$$9y^{10} + 45y^8 + 90y^6 + 90y^4 + 5y^2 - 3 = -9y^2 - 40y^4 + 6.$$

Uniting terms, $9(y^2 + y)^5 = 9 \therefore y^2 + y = 1$. Hence

$$y = -\frac{1}{2} \pm \sqrt{\frac{5}{4}} = -\frac{1 \pm \sqrt{5}}{2}.$$

SECTION IV.

TRINOMIAL EQUATIONS.

399. A Trinomial Equation is one that contains only two powers of an unknown quantity, excluding the zero power.

Such an equation may always be reduced to the form $x^n + 2px^m = q$, by a method similar to that pursued in (378). Transposing, we obtain, $x^n + 2px^m - q = 0$, in which equation, the second member being 0, the first is a *trinomial*, whence the name.

400. We cannot solve *all* trinomial equations; we can, in general, solve only those in which the exponent of the lower power of the unknown quantity is *one-half* the exponent of the higher power. Such equations will be of the form $x^{2n} + 2px^n = q$.

401. To Deduce a Rule for solving an equation of the form. $x^{2n} + 2px^n = q$. (1).

If, in this equation, we consider x^n as the unknown quantity, the equation may be written $(x^n)^2 + 2p(x^n) = q$ (2), a complete equation of the second degree. Solving, with reference to x^n , we obtain, $x^n = -p \pm \sqrt{q + p^2}$. Hence (383) $x = \pm \sqrt[n]{-p \pm \sqrt{q + p^2}}$, or $\sqrt[n]{-p \pm \sqrt{q + p^2}}$, as n is even or odd. Hence the

RULE.—Any equation being reduced to the form $x^{2n} + 2px^n = q$, to find the values of x : If n be even, take plus or minus the n th root of half the coefficient of the lower power of x , with its sign changed, increased or diminished by the square root of the second member, plus the square of this half coefficient. If n be odd, take simply the n th root of this quantity.

402. If n is even, the application of the rule gives four roots, i. e., two sets, the roots in each set being numerically equal, with contrary signs. The sign of $-p \pm \sqrt{q + p^2}$

being determined, as in (387), we can tell the number of real and of imaginary roots. If n is odd, we obtain but two roots, one having the sign of $-p + \sqrt{q + p^2}$, the other having the sign of $-p - \sqrt{q + p^2}$.

403. We have found that every equation of the first degree, containing but one unknown quantity, has but one root, and every equation of the second degree has two roots, and only two. We shall find, similarly, that every equation of the m th degree has m roots, and no more. Hence the equation $x^{2n} + 2px^m = q$, has $2n$ roots, of which we can find not more than 4 by (401); the others we find by methods treated in the higher Algebra.

EXAMPLES.

$$1. 14 + 4x^2 - \frac{x^2 + 7}{x^2 - 7} = 3x^2 + \frac{9 + 4x^2}{3}$$

Clearing of fractions,

$$12x^4 - 42x^2 - 294 - 3x^2 - 21 = 9x^4 - 63x^2 + 4x^4 - 19x^2 - 63;$$

uniting terms, $x^4 - 37x^2 = -252$

$$x = \pm \sqrt{\frac{37}{2} \pm \sqrt{-252 + \frac{1369}{4}}} = \pm \sqrt{\frac{37}{2} \pm \frac{19}{2}} = 28 \text{ or } 9$$

$$= \pm \sqrt{28} \text{ and } \pm 3.$$

$$2. \frac{8x^3}{x^2 + 2} - 6 = \frac{20}{3x^2}. \quad x = \sqrt[3]{10} \text{ and } -\frac{1}{3}\sqrt[3]{18}.$$

$$3. 2x^4 - 7x^2 = 99.$$

$$4. x^3\sqrt{x} - 4x\sqrt{x^3} + 3 = 0. \text{ Reducing to the form}$$

$$x^{3n} + 2px^m = q, \left(x^{\frac{5}{3}}\right)^2 - 4\left(x^{\frac{5}{3}}\right) = -3 \therefore x = 1 \text{ and } x = 1.93318$$

$$5. \frac{3x^3 - 10}{5(x^2 - 5)} = \frac{1}{20}.$$

$$6. x^3 - 8x^2 = 513.$$

$$\text{Ans. } x = 3 \text{ or } -\sqrt[3]{19}.$$

$$7. x^4 - 35x^2 + 216 = 0.$$

$$\text{Ans. } x = 2 \text{ or } 3.$$

$$8. x^{14} + .953x^7 = .047.$$

$$\text{Ans. } x = -1 \text{ or } 0.6461.$$

9. $x+4\sqrt{x}=21$.

Ans. $x=9$ or 49 .

In verifying this equation, $\sqrt{x}$ must be taken $+3$, unless we understand the double sign before the radical, when $+3$ may be used with the upper, and -3 with the lower, sign. Why is this the case? Would $\sqrt{x}=-7$ satisfy the equation as it stands? Why?

10. $3ax^{\frac{m}{2n}} = -9ax^{\frac{m}{2n}} - \frac{8a}{9} + 27ax^{\frac{m}{n}} - \frac{28a}{9}$.

Ans. $x = \left(\frac{2}{3}\right)^{\frac{2n}{m}}$ or $\left(-\frac{2}{9}\right)^{\frac{2n}{m}}$.

404. The rule (401) may also be applied to the solution of any equation from which we wish to find the value of any algebraic expression occurring therein with only two exponents, of which the higher is double the lower.

1. Take the equation,

$$12x + \frac{12x}{x} - 210 = 5 - 3x^2 - 6a - 3\frac{a^2}{x^2} + 205.$$

We may write the equation,

$$12\left(x + \frac{a}{x}\right) - 210 = 5 - 3\left(x + \frac{a}{x}\right)^2 + 205.$$

Transposing,

$$3\left(x + \frac{a}{x}\right)^2 + 12\left(x + \frac{a}{x}\right) = 420, \text{ or } \left(x + \frac{a}{x}\right)^2 + 4\left(x + \frac{a}{x}\right) = 140.$$

Solving, $x + \frac{a}{x} = 10$, or -14 . Hence,

$x^2 - 10x = -a$, and $x = 5 \pm \sqrt{25 - a}$, $x^2 + 14x = -a$, and $x = -7 \pm \sqrt{49 - a}$.

2. $\sqrt{x+12} + \sqrt[4]{x+12} = 6$.

Ans. $x=4$ or 69 .

3. $\frac{\sqrt[5]{x-a}}{\sqrt[5]{x-a}+1} + \frac{\sqrt[5]{x-a}+1}{\sqrt[5]{x-a}} = \frac{18}{6}$.

4. $x^2 + 16 + \frac{64}{x^2} + x = 42 - \frac{8}{x}$.

Ans. $x=4$, or 2 , and $\frac{-7 \pm \sqrt{17}}{2}$.

$$5. \sqrt[10]{3x-14} + 6\sqrt[5]{3x-14} - 7 = 4\sqrt[10]{3x-14} - 3\sqrt[5]{3x-14} - 1.$$

Verify.

6. $x^2 + 2ax^2 + 5a^2x + 4a^2 = 0$; multiplying through by x ,
 $x^3 + 2ax^3 + 5a^2x^2 + 4a^2x = 0$; or $(x^2 + ax)^2 + 4a^2(x^2 + ax) = 0$.
 Whence, $x^2 + ax = -4a^2$, or 0.

7. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$; the equation may be written, $(x^2 - x)^2 - 8(x^2 - x) + 12 = 0$.

$$8. x^2 - 6x^2 + 3x = 18.$$

$$\text{Ans. } x = \pm \sqrt{-3}.$$

405. An equation may often be reduced to the form now under discussion, by transposing all the terms to the first member, arranging with reference to the unknown quantity, and extracting the square root, making the highest exponent even, if necessary, by multiplying through by the unknown quantity, and the corresponding coefficient a perfect square. If the root is a factor of the remainder, or part of the remainder, this root will be the quantity with reference to which the given equation may be solved.

9. Take the equation $x^4 - 14x^3 + 56x^2 - 49x = 60$.

$$\begin{array}{r} x^4 - 14x^3 + 56x^2 - 49x - 60 \quad |x^2 - 7x \\ \underline{x^4 - 14x^3 + 49x^2} \\ 7x^2 - 49x - 60. \end{array} \quad \begin{array}{l} \text{Proceeding as directed,} \\ \text{we find that the partial} \\ \text{square root } x^2 - 7x, \text{ is a} \end{array}$$

factor of the first part of the remainder. Arranging the given equation then, with reference to $x^2 - 7x$, we have,
 $(x^2 - 7x)^2 + 7(x^2 - 7x) - 60 = 0$,

$$x^2 - 7x = -\frac{7}{2} \pm \sqrt{60 + \frac{49}{4}} = -\frac{7}{2} \pm \frac{17}{2} = 5 \text{ or } -12,$$

$$x^2 - 7x = 5, \quad x = \frac{7}{2} \pm \sqrt{\frac{69}{4}} = \frac{1}{2} (7 \pm \sqrt{69}),$$

$$x^2 - 7x = -12, \quad x = \frac{7}{2} \pm \sqrt{\frac{1}{4}} = 4 \text{ or } 3.$$

With equations of the fourth degree it is unnecessary to go further than two terms, in extracting the root.

$$10. x^2 + 5x + 4 = 5(x^2 + 5x + 28)^{\frac{1}{2}}.$$

$$x = 4 \text{ or } -9, \text{ or } -\frac{1}{2}(5 \pm \sqrt{-51}).$$

$$11. \sqrt{1+x-x^2}-2(1+x-x^2)=\frac{1}{9}.$$

$$x=\frac{1}{2}\left(\frac{3\pm\sqrt{41}}{3}\right), \text{ or } \frac{1}{2}\left(\frac{3\pm2\sqrt{11}}{3}\right).$$

$$12. x^6+20x^3=69.$$

$$x=\sqrt[3]{3} \text{ or } -\sqrt[3]{23}.$$

$$13. x^4+x^3-4ax^2+ax+a^2=0. \text{ Dividing by } x^2,$$

$$x^2+x-4a+\frac{a}{x}+\frac{a^2}{x^2}=0, \text{ or } \left(x+\frac{a}{x}\right)^2+\left(x+\frac{a}{x}\right)=6a.$$

$$14. \quad x-1=2+\frac{2}{\sqrt{x}}. \text{ Transposing, } x-3=\frac{2}{\sqrt{x}};$$

$$\text{squaring, } x^2-6x+9=\frac{4}{x}, \text{ or } x^3-6x^2+9x=4; \text{ whence,}$$

$x^4-6x^3+9x^2-4x=0$. Adding x^3+x to both members,
 $x^4-6x^3+10x^2-3x=x^3+x$, or $(x^2-3x)^2+(x^2-3x)=x^3+x$.
 $x^2-3x=x$ or $-x-1$. Taking $x^2-3x=x$, $x=4$;
 taking $x^2-3x=-x-1$, $x^2-2x=-1$. $\therefore x=1$.

$$15. x^4-2x^3+x=132. \text{ This may be written}$$

$$(x^2-x)^2-(x^2-x)=132.$$

$$16. x^4+\frac{17}{2}x^3=34x+16. \text{ Adding } \frac{289}{16}x^2 \text{ to both members,}$$

$$x^4+\frac{17}{2}x^3+\frac{289}{16}x^2=\frac{289}{16}x^2+34x+16, \quad x=-\frac{1}{2} \text{ or } -8.$$

406. It may often be useful to remember that if a polynomial equals zero, the product of its factors equals zero; and hence we may place each factor separately equal to zero. This is merely an extension of a principle already (388) employed. For example, let $6x^4-2ax^3+14x^2+18x^3-3ax^2+42x-a^2x+7a=0$. The factors of the first member are, $3x^2-ax+7$ and $2x^3+6x+a$, placing either of which equal to 0, we obtain an equation that we can solve exactly. This is one more illustration of the importance of practical skill in factoring.

407. Double Radicals.—The solution of trinomial equations of the fourth degree frequently gives rise to ex-

pressions of the form $\sqrt{a \pm \sqrt{b}}$, called *Double Radicals*, a and b being numerical or literal, positive or negative, entire or fractional. As these expressions require the extraction of the square root of radicals, we always reduce them, when such reduction is possible, to the form $a' \pm \sqrt{b'}$ or $\sqrt{a'} \pm \sqrt{b'}$. These forms are more convenient, as they indicate merely the extraction of roots of rational quantities. $\sqrt{a \pm \sqrt{b}}$ may represent any double radical deduced from an equation of the fourth degree, for the most general form of such a radical would be $\sqrt{a \pm f \sqrt{b''}}$, which may be reduced to the above form by placing f under the second radical, thus: $\sqrt{a + \sqrt{f^2 b''}}$ or $\sqrt{a + \sqrt{b}}$ (I).

408. To Deduce a Formula for reducing any Radical of the form $\sqrt{a \pm \sqrt{b}}$ to the form $\sqrt{a'} \pm \sqrt{b'}$.

Let $\sqrt{a + \sqrt{b}} = p + q$ (1) $\left\{ \begin{array}{l} p \text{ and } q \text{ being quantities the} \\ \sqrt{a - \sqrt{b}} = p - q \text{ (2)} \end{array} \right\}$ values of which are to be determined so that they may fulfill these conditions. Squaring the members of (1) and (2), we obtain

$$a + \sqrt{b} = p^2 + 2pq + q^2 \quad (3)$$

$$a - \sqrt{b} = p^2 - 2pq + q^2 \quad (4)$$

Multiplying (1) by (2), member by member,

$$\sqrt{a^2 - b} = p^2 - q^2 = c \quad (5), \text{ placing } \sqrt{a^2 - b} = c.$$

Adding (3) and (4), member to member, we find $2a = 2(p^2 + q^2)$ or $a = p^2 + q^2$ (6). Adding (5) and (6), member to member,

we find $2p^2 = a + c \therefore p = \pm \sqrt{\frac{a+c}{2}}$. Subtracting (5) from

(6), member from member, $2q^2 = a - c \therefore q = \pm \sqrt{\frac{a-c}{2}}$.

$$\text{Hence, } p + q = \sqrt{a + \sqrt{b}} = \pm \left(\sqrt{\frac{a+c}{2}} + \sqrt{\frac{a-c}{2}} \right) \quad (7)$$

$$p - q = \sqrt{a - \sqrt{b}} = \pm \left(\sqrt{\frac{a+c}{2}} - \sqrt{\frac{a-c}{2}} \right) \quad (8)$$

If a^2-b is a perfect square, c will be rational, and (7) and (8) are the formulæ required. If a^2-b is not a perfect square, c is a radical, and the transformation is of no service. Before applying (7) and (8), therefore, we must satisfy ourselves that a^2-b is a perfect square.

EXAMPLES.

Reduce the following radicals to their simplest forms:

1. $\sqrt{8+2\sqrt{15}} - \sqrt{8-2\sqrt{15}}$.

Reducing to the form (1), we obtain

$$\sqrt{8+2\sqrt{60}} - \sqrt{8-2\sqrt{60}}. \quad \sqrt{a^2-b} = \sqrt{64-60} = 2.$$

Hence the formula may be used,

$$\frac{a+c}{2} = \frac{8+2}{2} = \frac{10}{2} = 5; \quad \frac{a-c}{2} = \frac{8-2}{2} = 3.$$

Hence we have $\pm(\sqrt{5} + \sqrt{3})$, from which we subtract $\pm(\sqrt{5} - \sqrt{3})$, obtaining $2\sqrt{3}$.

2. $\sqrt{ab+4c^2-d^2+2\sqrt{4abc^2-abd^2}}$.

In (I), $a^2 = a^2b^2 + 8abc^2 - 2abd^2 - 8d^2c^2 + d^4 + 16c^4$

$$b = \frac{16abc^2 - 4abd^2}{a^2 - b = a^2b^2 - 8abc^2 + 2abd^2 - 8d^2c^2 + 16c^4 + d^4}$$

$$\sqrt{a^2-b} = 4c^2 - \frac{d^2-ab=c}{a=4c^2 - \frac{d^2+ab=a}{a+c=8c^2 - 2d^2} \quad a-c=2ab \therefore}$$

$$\frac{a+c}{2} = 4c^2 - d^2; \quad \frac{a-c}{2} = ab.$$

$$\sqrt{ab+4c^2-d^2+2\sqrt{4abc^2-abd^2}} = \pm(\sqrt{4c^2-d^2} + \sqrt{ab}).$$

3. $\sqrt{11+2\sqrt{30}} + \sqrt{11-2\sqrt{30}}$. Ans. $2\sqrt{6}$.

4. $\sqrt{18+8\sqrt{5}} - \sqrt{18-8\sqrt{5}}$. Ans. $4\sqrt{2}$.

5. $\sqrt{3-4\sqrt{-1}}$. Ans. $2-\sqrt{-1}$.

6. $\sqrt{bc+2b\sqrt{bc-b^3}} - \sqrt{bc-2b\sqrt{bc-b^3}}$. Ans. $\pm 2b$.

7. $\sqrt{80+16\sqrt{-64}}$.

$$8. \sqrt{30-6\sqrt{39+18}}. \quad \text{Ans. } \pm(\sqrt{39}-3).$$

$$9. \sqrt{6xy+4z^2-16v^2+2\sqrt{24xyz^2-96xyv^2}}. \\ \text{Ans. } \pm(2\sqrt{z^2-4v^2}+\sqrt{6xy}).$$

$$10. \sqrt{4+3\sqrt{-20}}+\sqrt{4-3\sqrt{-20}}. \quad \text{Ans. } 6.$$

The general discussion of the double radical, $\sqrt[3]{a \pm \sqrt[3]{b}}$, is not of sufficient utility or importance to justify our giving it space here.

SECTION V.

SIMULTANEOUS EQUATIONS OF HIGHER DEGREES.

409. Simultaneous equations of higher degrees than the second may often be solved by the rules and expedients already discussed in connection with equations of the second degree. The principal devices resorted to are those discussed in **A, II, III, IV**, and a few that cannot be conveniently classified. As we cannot here enter into a complete discussion of the subject, we shall illustrate, by a few examples, the most important methods of solution; examples that will give the student an idea of the ways of operating; practice, assisted by ingenuity and close observation, will render him skillful in the treatment of the different cases that may arise.

EXAMPLES.

1. $x+y=a$ (1); $x^2+y^2=c$ (2). Here we may divide (2) by (1), member by member, obtaining $x^2-xy+y^2=\frac{c}{a}$, an equation of the second degree, in which we may substitute the value of x or y from (1). Or, place $x=s+z$, $y=s-z$, since (1) and (2) are symmetrical. Substituting in (1), $2s=a$; (2), $2s^2+6sz^2=c$, and $z=\pm\sqrt{\frac{c-2s^2}{6s}}$, whence,

$x=s\pm\sqrt{\frac{c-2s^2}{6s}}$, and $y=s\pm\sqrt{\frac{c-2s^2}{6s}}$. Substituting for s its value, $\frac{a}{2}$, we obtain,

$$x=\frac{a}{2}\pm\sqrt{\frac{c-2\frac{a^2}{4}}{3a}}=\frac{a}{2}\pm\sqrt{\frac{4c-a^2}{12a}}, \text{ and } y=\frac{a}{2}\mp\sqrt{\frac{4c-a^2}{12a}}.$$

2. $x+y=6$, $x^4+y^4=272$.

From (1) $x^4+4x^2y+6x^2y^2+4xy^3+y^4=1296$.

From (2) $x^4+y^4=272$.

Hence, by subtraction, $4x^2y+6x^2y^2+4xy^3=1024$,
or, $2xy(2x^2+3xy+2y^2)=1024$ (3)

From (1), $x^2+2xy+y^2=36$; $2xy(2x^2+4xy+2y^2)=144xy$ (4)

Subtracting (3) from (4), member from member,

$$2x^2y^2=144xy-1024.$$

Solving with reference to xy , $xy=64$ (5), or $xy=8$ (6).

Combining (5) and (6) with (1), we find $y=2$, or 4, and $3\pm\sqrt{-55}$; $x=4$ or 2, and $3\mp\sqrt{-55}$.

3. $x+y+\sqrt{x+y}=12$ (1) Solve (1) with regard to $x+y$
 $x^2+y^2=189$ (2) and combine the result with

(2), as in Example 1. Ans. $x=5$ or 4
 $y=4$ or 5.

4. $xy+xy^2=12$ (1) We may write (1) $xy(1+y)=12$ (3)
 $x+xy^2=18$ (2) and (2) $x(1+y^2)=18$ (4)

Dividing (4) by (3), member by member,

$$\frac{1}{y}(1-y+y^2)=\frac{3}{2} \text{ or } 2-2y+2y^2=3y; \quad y^2-2.5y=-1$$

$\therefore y=2$ or 0.5. Substituting in (4), $x=2$ or 16.

5. $x^2y+xy^2=6$ Ans. $x=2$ or 1
 $x^2y^2+x^2y^2=12$ $y=1$ or 2.

6. $x^2y+xy^2=30$ Ans. $x=2$ or 3
 $x^2+y^2=35$ $y=3$ or 2.

7. $x^2y^2+xy^2+xy+x^2y^2+2x^2y^2=600$ (1)
 $x+y+y^2=14$ (2)

We may write (1): $x^2y^2+(y^2+2y)x^2y^2+xy^2+xy=600$
 $x^2y^2(1+y^2+2y)+xy(1+y)=600$
 $x^2y^2(1+y)^2+xy(1+y)=600$

Solving with respect to $xy(1+y)$, we have,

$$xy(1+y) = -\frac{1}{2} \pm \sqrt{\frac{2401}{4}} = -\frac{1}{2} \pm \frac{49}{2} = 24 \text{ or } -25.$$

Taking the positive value, $x = \frac{24}{y+y^2}$; which, substituted in (2), gives $(y^2+y)^2 - 14(y^2+y) = -24$;

whence, $y^2+y=12$ or 2 , and $x = \frac{24}{12}$ or $\frac{24}{2} = 2$ or 12 .

$$\text{Taking } y^2+y=12; y = -\frac{1}{2} \pm \sqrt{\frac{49}{4}} = -\frac{1}{2} \pm \frac{7}{2} = 3 \text{ or } -4.$$

$$\text{Taking } y^2+y=2; y = -\frac{1}{2} \pm \sqrt{\frac{9}{4}} = -\frac{1}{2} \pm \frac{3}{2} = 1 \text{ or } -2.$$

If we take the second value of x , viz.: $-\frac{25}{y^2+y}$, we obtain, from (2), $(y^2+y)^2 - 14(y^2+y) = 25$;

$$y^2+y = +7 \pm \sqrt{25+49} = 74; = +7 \pm \sqrt{74};$$

$$y = -\frac{1}{2} \pm \sqrt{\frac{29}{4}} + \sqrt{74} = -\frac{1}{2}(1 \pm \sqrt{29 \pm 4\sqrt{74}})$$

$$\text{and } x = \frac{25}{7 \pm \sqrt{74}}$$

$$8. x^2 + \frac{x^2}{y^2} + y^2 = 84 \quad (1); \quad x + \frac{x^2}{y} + y = 14 \quad (2). \quad \text{Dividing (1)}$$

by (2), member by member, we obtain, $\frac{x^2}{y} + y - x = 6 \quad (3).$

$\frac{x^2}{y} + y + x = 14.$ Adding (2) and (3), member to member,

we obtain $2\left(\frac{x^2}{y} + y\right) = 20 \therefore \frac{x^2}{y} + y = 10.$ Substituting this

value for $\frac{x^2}{y} + y$, in (2), we find $x + 10 = 14 \therefore x = 4. \therefore$

$\frac{x^2}{y} + y = 10, x^2 + y^2 = 10y \therefore 16 = 10y - y^2$, or $y^2 - 10y = -16$;

whence $y = 5 \pm \sqrt{9} = 8$ or 2 .

$$9. 2(x^2 - y^2)(x - y) = 32xy \quad (1); \quad \frac{x^4 - y^4}{40} = \frac{160x^2y^2}{x^2 - y^2} \quad (2).$$

Clearing of fractions and reducing, we have,

$$(x^2 - y^2)(x - y) = 16xy \quad (3); \quad (x^4 - y^4)(x^2 - y^2) = 640x^2y^2 \quad (4).$$

Dividing (4) by (3), member by member,

$$\frac{x^2 + x^2y + xy^2 + y^2}{x^2 - x^2y - xy^2 - y^2} = \frac{40xy}{16xy} \quad (5)$$

$$\text{From (1), } \frac{x^2 - x^2y - xy^2 - y^2}{x^2y + xy^2} = \frac{16xy}{12xy} \quad (1)$$

$$\text{Hence } \frac{x^2y + xy^2}{xy(x+y)} = \frac{12xy}{12xy}$$

Whence, $(x+y)=12$, and $x=12-y$, and $x^2+2xy+y^2=144$, $x^2+y^2=144-2xy$ (6).

From (3) and (4), $(x^2+y^2)(x+y)=40xy$ (5); $(x^2+y^2)12=40xy$ (7). Combining (6) and (7), we find $xy=27 \therefore x=\frac{27}{y}$.

Whence, $27+y^2=12y$, $y^2-12y=-27$, $y=6\pm\sqrt{9}=6\pm 3=9$ or 3, and $x=3$ or 9.

$$10. \frac{(x+y)(xy+1)}{x^4y^3+x^2y^4+x^3+y^2} = \frac{18xy}{208x^2y^2} \quad (1)$$

Factoring in (2), and dividing each member by x^2y^2 , and dividing each member of (1) by xy ,

$$x + \frac{1}{x} + y + \frac{1}{y} = 18; \quad x^2 + \frac{1}{x^2} + y^2 + \frac{1}{y^2} = 208.$$

Let $x + \frac{1}{x} = s$ and $y + \frac{1}{y} = v$; whence, $v+s=18$;

$v^2+s^2=212$; whence, $v=14$ or 4; $s=4$ or 14.

$x^2+1=4x$ or $14x$; $y^2+1=14y$ or $4y$; whence,

$x=2\pm\sqrt{3}$, or $7\pm 4\sqrt{3}$; $y=7\pm 4\sqrt{3}$, or $2\pm\sqrt{3}$.

PROBLEMS.

1. Find two numbers, such that the sum of the products obtained by multiplying them by a and b respectively may be equal to $2s$, and the product of the numbers may be equal to p .

Let x represent the first, and y the second, number.

From the conditions, $ax+by=2s \dots (1)$. $xy=p \dots (2)$.

From (1), $y = \frac{2s-ax}{b}$; substituting this value in (2), $\frac{2sx-ax^2}{b} = p$, or

$$ax^2-2sx=-pb; \text{ whence } x = \frac{s}{a} \pm \frac{\sqrt{s^2-apb}}{a} = \frac{s \pm \sqrt{s^2-apb}}{a}.$$

From (1), $x = \frac{2s-by}{a}$, whence, (2), $2sy-by^2=ap \therefore y = \frac{s \pm \sqrt{s^2-bpa}}{b}$.

There will, in general, be two solutions to the problem, the sign being

governed in each case by the sign of s , since $s > \sqrt{s^2 - apb}$ and $s > \sqrt{s^2 - bpa}$. If $s^2 = apb$, the two values of x reduce to $\frac{s}{a}$, and the two values of y reduce to $\frac{s}{b}$; and if $a = b$, the problem has practically but one solution, since the two values of x and the two values of y reduce to the same. If $s^2 < apb$, both values of x and y become imaginary, i. e., the solution of the problem, if understood in the strict arithmetical sense, is impossible.

2. The sum of two numbers is 4, and the sum of their fourth powers is 82. What are the numbers?

Ans. 3 and 1.

3. In Delton are two squares, the difference in area being q^2 . If each side of the first were a times its present length, and each side of the second b times its present length, the first would be longer by d than the second. Required, the length of each.

Let x represent the number of units in a side of the first.

Let y represent the number of units in a side of the second.

From the conditions, $x^2 - y^2 = q^2$ (1), $ax - by = d$ (2). From (2), $y = \frac{-d + ax}{b}$; which, substituted in (1), gives $x^2 - \frac{a^2x^2 - 2adx + d^2}{b^2} = q^2$;

$$b^2x^2 - a^2x^2 + 2adx - d^2 = b^2q^2 \quad (3), \quad x^2(b^2 - a^2) + 2adx = d^2 + b^2q^2,$$

$$x^2 - \frac{2ad}{a^2 - b^2}x = \frac{-b^2q^2 - d^2}{a^2 - b^2},$$

$$\begin{aligned} x &= \frac{ad}{a^2 - b^2} \pm \sqrt{\frac{-a^2b^2q^2 - a^2d^2 + b^4q^2 + b^2d^2 + a^2d^2}{(a^2 - b^2)^2}} \\ &= \frac{ad}{a^2 - b^2} \pm b \sqrt{\frac{d^2 - (a^2 - b^2)q^2}{(a^2 - b^2)^2}} = \frac{ad \pm b\sqrt{d^2 - (a^2 - b^2)q^2}}{a^2 - b^2} \end{aligned}$$

$$\text{In a similar way would be obtained, } y = \frac{bd \pm a\sqrt{d^2 - (a^2 - b^2)q^2}}{a^2 - b^2}.$$

4. A number is expressed by three digits, the square of the second of which equals the product of the first and third. The number itself is to the sum of its digits as 248 is to 14; and if 297 be added to it, the result will be expressed by the same digits reversed. What is the number?

Ans. 124.

5. The fore-wheel of a carriage makes 6 revolutions more than the hind-wheel in going 120 M; but if the circumference of each wheel were increased by 1 M, the fore-wheel would make only 4 revolutions more than the hind-wheel in the 120 M. Required, the circumference of each wheel.

Ans. 4 and 5 M, respectively.

6. A broker sold 80 shares of S. V. stock and 100 shares of W. C. stock, for \$6500; he sold 6 shares more of S. V. stock for \$500, than he did of W. C. stock for \$100. Required, the price of each.

Ans. \$50 and \$25 per share, respectively.

7. The joint stock of two partners was \$208,000; Thomson's money was in trade 9 months, and Bender's 6 months. When they shared stock and profits, Thomson received \$114,000, and Bender \$126,000. Required, the stock of each.

Ans. \$96,000 and \$112,000, respectively.

8. A certain cistern, holding a gallons, was exactly full, and was to be emptied by two engines. No. 1 pumped for $\frac{2}{3}$ of the time that it would have taken No. 2 to empty the cistern, and at the end of this time, was relieved by No. 2, which completed the work. Had the two engines worked together, they would have emptied the cistern 6 hours sooner, and No. 1 would have pumped out $\frac{2}{3}$ of the number of gallons left for No. 2. Required, the time in which each engine would, alone, complete the work.

Let $5x$ = the number of hours required by engine No. 1.

Let $5y$ = the number of hours required by engine No. 2.

In one hour, then, No. 1 pumped out $\frac{a}{5x}$ gallons, and in $\frac{3}{5}$ of $5y$, or $3y$ hours, it must have pumped out $\frac{3ay}{5x}$ gallons. No. 2 pumped out the remaining $(a - \frac{3ay}{5x})$ gallons, in $(5y - \frac{3y^2}{x})$ hours. For, since it would pump out all the water in $5y$ hours, it pumps out the part $\frac{a}{5y}$ in 1 hour, and would therefore require as many hours to pump $(a - \frac{3ay}{5x})$ gallons as

is expressed by $(a - \frac{3ay}{5x}) \div \frac{a}{5y} = (5y - \frac{3y^2}{x})$. In one hour the two engines, working together, would pump $\frac{a}{5x} + \frac{a}{5y} = a \frac{x+y}{5xy}$ gallons. They therefore would empty the cistern in $\frac{5xy}{x+y}$ hours; and in $\frac{5xy}{x+y}$ hours No. 1 pumps $\frac{5xy}{x+y} \times \frac{a}{5x} = \frac{ay}{x+y}$ gallons. We have, then,

$$\frac{5xy}{x+y} = 3y + (5y - \frac{3y^2}{x}) - 6 \quad (1), \quad \frac{ay}{x+y} = \frac{2}{3} (a - \frac{3ay}{5x}) \quad (2).$$

Solving (1) and (2), we find that the engines would empty the cistern in 15 and 10 hours, respectively.

9. Equal monthly rents are obtained for two pieces of property, that are together worth \$60,000. If the second were to pay the same per cent. on its value as the first, the rent of it would be \$481.66 $\frac{2}{3}$ per month; while if the first were to pay the same per cent. on its value as the second, the rent of it would be \$281.66 $\frac{2}{3}$ per month. Required, the value of each piece of property, and the percentage of income realized from each.

Ans. \$26,000; \$34,000; 17% per annum; 13% per annum.

10. A and B engage to reap a field for 90 shillings, and as A alone could reap it in 9 days, they promise to finish it in 5 days. Obligated to call in C, an inferior workman, to assist them during the last 2 days, B receives 3s. 9d. less than he would otherwise have received. In what time could B or C alone reap the field.

Ans. 15 and 18 days respectively.

11. A number is expressed by two digits; the difference of these digits is 1, and if their sum be increased by the cube of itself, the result will be 30. Required, the number.

Ans. 21 or 12.

12. Find two numbers, such that the difference of their squares shall be $\frac{1}{14}$ of the difference of their cubes, and $\frac{1}{18}$ of the sum of their cubes.

Ans. 6 and 12.

CHAPTER VIII.

SECTION I.

ZERO—INFINITY—INDETERMINATES.

410. The term **Zero** (0) is used in Algebra in two senses:

1°. The strictly *Arithmetical* sense of *no quantity, nothing, naught*.

2°. The strictly *Algebraic* sense of *a quantity less than any assignable quantity*, or, as often named, an *Infinitesimal*.

We shall, in general, use the term in its strictly *algebraic* sense. The symbol 0 should then be called *zero*, and not *naught*, for *naught* means simply *nothing*, and is therefore the proper name for the symbol only when used in the *arithmetical* sense.

Zero does not mean the smallest possible quantity, but simply a quantity of which the value is less than any assignable limit. Hence, as quantities may be less than any assignable quantity, and yet not be equal to each other, all zeroes are not equal; they may, in fact have *any relation whatever to each other*, as we shall explain hereafter.

411. Infinity (∞) is a quantity *greater than any assignable quantity*; a quantity is said to be *infinite* when it is greater than any assignable limit. All infinite quantities are not equal; they may, as will be hereafter shown, have *any relations whatever to each other*.

412. A finite quantity is a quantity that is neither zero nor infinity.

413. Proposition I.—*A finite quantity, divided by infinity, is zero.*

In the fraction $\frac{a}{x}$, suppose that, a remaining always the same, x continually increases. Since the numerator remains the same, while the denominator becomes greater and greater, the value of the fraction must become smaller and smaller. When the value of x is very great, the value of the fraction must be very small. Finally, when x becomes greater than any assignable quantity, the value of the fraction must become less than any assignable quantity; i. e., it must become zero, or, $\frac{a}{\infty} = 0$ (1). As a may represent any finite quantity, the truth of the proposition is established.

Corollary. — Clearing the equation (1) of fractions, $a=0 \cdot \infty$, that is, $0 \cdot \infty$ may represent any finite quantity.

Proposition II.—Zero, divided by any finite quantity, is zero.

Proposition III.—Any finite quantity, divided by zero, is infinity.

Suppose that, in the fraction $\frac{a}{x}$, a remaining always the same, x continually diminishes. The value of the fraction obviously increases; and the smaller the value of x , the greater will be the value of the fraction. When x becomes very small, the value of the fraction becomes very great. Finally, when x becomes less than any assignable quantity, the value of the fraction becomes greater than any assignable quantity, i. e., $\frac{a}{0} = \infty$. Since a may represent any finite quantity whatever, the truth of the proposition is established.

Proposition IV.—Infinity, divided by any finite quantity, is infinity.

Proposition V.—Zero and infinity are reciprocals of each other.

$$\frac{a}{0} = \frac{1}{\frac{0}{a}} = \frac{1}{0} = \infty. \quad \text{Similarly } 0 = \frac{0}{a} = \frac{1}{\frac{a}{0}} = \frac{1}{\infty}.$$

Proposition VI.—*Zero multiplied by any finite quantity equals zero.*

Since zero is less than any assignable quantity, no matter how many times it be added to itself, so long as the number of times is finite, the result must be less than any assignable quantity, i. e., zero.

Proposition VII.—*Infinity multiplied by any finite quantity is infinite.*

Proposition VIII.—*Zero divided by zero (taken without reference to its origin), may represent any quantity whatever.*

We have already shown that 0 means simply a quantity less than any assignable quantity; and as there may be any number of quantities less than any assignable quantity, 0 may represent any number of quantities, having any relation whatever to each other. Hence, $\frac{0}{0}$ conveys no more

definite an idea than $\frac{x}{y}$, as the numerator and denominator may have any relation to each other, and therefore the fraction may have an infinite number of values. Hence, $\frac{0}{0}$ may represent any quantity whatever.

Proposition IX.—*Infinity divided by infinity (taken without reference to its origin), may represent any quantity whatever.*

$$\frac{\infty}{\infty} = \frac{\frac{1}{0}}{\frac{1}{0}} = \frac{1}{0} \times \frac{0}{1} = \frac{0}{0}. \quad \text{Hence, (Prop. VII), } \frac{\infty}{\infty} \text{ may}$$

represent any quantity whatever.

ILLUSTRATION I. $\frac{0}{0}$ may represent any quantity whatever.

Suppose, s , b , and l representing any quantities whatever, $\frac{s}{b} = l$. Since dividing both terms of this fraction by any quantity will not alter the value of the fraction, we may divide both s and b by 1000, 10,000, etc., successively,

and the value of the fraction will always be l . If, now, the division be continued until the numerator and denominator are both less than any assignable quantity, the value of the fraction will still be l , i. e., $\frac{0}{0} = l$. As l may represent any quantity whatever, let it represent 1; then $\frac{0}{0} = 1$, and the zeroes in this case are equal; if $l=3$, $\frac{0}{0} = 3$, and one zero is three times as great as the other.

Suppose l to represent the relation between the sides, s and b , of a rectangle, in which one side is 7 times as long as the next one. No matter how small the rectangle, while it preserves its shape $\frac{s}{b}$ will always be precisely equal to 7. Hence, when the rectangle becomes smaller than any assignable quantity, $\frac{s}{b} = \frac{0}{0} = 7$. Pursuing a similar course with different values and substitutions, we can easily see how $\frac{0}{0}$ may represent any quantity whatever.

ILLUSTRATION II. $\frac{\infty}{\infty}$ may represent any quantity whatever.

Suppose the numbers $2 + 2 + 2 + 2 + 2 + \dots$

$10 + 10 + 10 + 10 + 10 + 10 + \dots$ be continued

until there is an infinite number of terms in each sum. No matter how far the numbers may be continued, while the two sets are continued to the same extent, the sum of the lower line will always be 5 times as great as the sum of the upper line. If, then, the sets be continued together until we have an infinite number of terms, the sum of each line will be infinite, but the sum in the upper line will be only $\frac{1}{5}$ of the sum in the lower. In this case, then, $\frac{\infty}{\infty}$ will represent 5, or $\frac{1}{5}$, according as we take the sum of the upper or of the lower set in the denominator. Instead of 2, we could substitute any number whatever, 1, 3, 4, 5, 6, 7, etc., and similarly, instead of 10, any number, 5, 7, 12, 3, 1, 6, etc. The lines being continued together until the number of terms is infinite, the ratio of the upper to the lower infinity would be $\frac{1}{5}$, $\frac{3}{7}$, $\frac{4}{12}$, $\frac{5}{3}$, 7, etc., according to the numbers substituted, and by making the ratio of the numbers substituted anything we please, we make $\frac{\infty}{\infty}$ represent anything we please.

Again, let a rectangle, of which s and b represent the sides, be six times as long as it is broad, whence $s:b=6$. If the rectangle grows larger, retaining its shape, $s:b$ will always be 6; finally, when the rectangle becomes greater than any assignable quantity, $\frac{s}{b}=\frac{\infty}{\infty}$, still equals 6. By assuming different ratios for the sides of the original rectangle, we can see how $\frac{\infty}{\infty}$ may be made to represent any quantity whatever.

The ratio $\frac{\infty}{0}$, or $\frac{0}{0}$, is not necessarily finite. For, let $\frac{z}{w}=\frac{1}{z}$. If, now, we assume z and w to decrease continually, $\frac{z}{w}=\frac{1}{z}$ will increase, and finally, when $z=0$ and $w=0$, we shall have $\frac{0}{0}=\frac{1}{0}=\infty$. If, on the other hand, we had supposed z and w to increase, until $z=\infty$ and $w=\infty$, we should have had $\frac{\infty}{\infty}=\frac{1}{\infty}=0$.

Hence we see that two quantities may be both 0, and yet one be infinitely greater than the other. Again, two quantities may both be infinite, and yet one be infinitely small in comparison with the other.

Proposition X.—*Infinity plus or minus a finite quantity is infinity.*

Proposition XI.—*A finite quantity plus or minus zero equals the original finite quantity.*

Proposition XII.—*Infinity minus infinity may have any value whatever.*

414. An Indeterminate quantity, or Indeterminate is a quantity that admits of an infinite number of values.

Such a quantity is called *indeterminate*, for the reason that we cannot *fix* or *determine* it, by giving it any particular value, as it has an infinite number of other values equally applicable.

415. Symbols of Indetermination. Since $\frac{0}{0}$ may represent an infinite number of quantities, it is generally employed to denote a quantity that has an infinite number of values, *i. e.*, an indeterminate. It is known generally as **The symbol of indetermination.** $\frac{\infty}{\infty}$ (413, IX) may also be used as a symbol of indetermination, but it may be re-

garded as a special form of

$$\frac{0}{0}, \text{ since } \frac{\infty}{\infty} = \frac{\frac{1}{0}}{\frac{1}{0}} = \frac{1}{0} \times \frac{0}{1} = \frac{0}{0}.$$

So $\infty - \infty$ and $0 \cdot \infty$ are symbols of indetermination.

We have found that $\frac{0}{0}$, $\frac{\infty}{\infty}$, etc., *when taken without reference to their origin*, may represent any quantity whatever. This reservation is very important, for we shall find that $\frac{0}{0}$ may arise from some special circumstances. A ratio may be *concealed* under the form $\frac{0}{0}$, in which case the fraction may not be indeterminate. Whenever $\frac{0}{0}$ is the *true* ratio, this ratio is indeterminate.

SECTION II.

VANISHING FRACTIONS.

416. A **Vanishing Fraction** is a fraction that becomes $\frac{0}{0}$, under some particular supposition.

417. The value $\frac{0}{0}$, obtained by making some supposition on the quantities in a fraction, is not always the true value of the fraction. To illustrate; take the fraction $\frac{y+10}{y+20}$. The value of this fraction, when $y=5$, is $\frac{5+10}{5+20} = \frac{15}{25} = \frac{3}{5}$. Now, multiply both terms of the original fraction by $y-5$. We obtain $\frac{(y+10)(y-5)}{(y+20)(y-5)}$; we have not altered the value of the fraction by this transformation, yet if, in the new fraction, we make $y=5$, we obtain $\frac{(5+10) \cdot 0}{(5+20) \cdot 0} = \frac{0}{0}$. The presence, in the terms, of this common factor, $(y-5)$, which,

under the particular hypothesis, reduces to 0, makes the fraction appear to be equal to $\frac{0}{0}$, i. e., to have an infinite number of values, whereas we know that it has but one, viz. $\frac{3}{5}$. The transformation has concealed the ratio, though it has not altered it. If, then, a fraction vanishes (i. e., reduces to $\frac{0}{0}$), under any supposition, we cannot conclude that it has an infinite number of values under that supposition, until we are sure that the vanishing does not result, as in this example, from the presence, in the terms of the fraction, of a common factor reducing to 0 under the particular supposition.

418. The Evaluation of a vanishing fraction is the process of finding the true value of the fraction under the supposition that makes it apparently $\frac{0}{0}$.

419. To Deduce a Rule for the evaluation of Vanishing Fractions.

Vanishing fractions are of two general classes:

I. Fractions of the form $\frac{Nx^m}{Dx^n}$ (vanishing when $x=0$).

II. Fractions of the form $\frac{N(y^m \pm z^m)}{D(y \pm z)}$ (vanishing when $y=\mp z$).

The symbols N, D, etc., represent any quantities whatever, monomial or polynomial, rational or irrational, etc.

CLASS I.

There may be three cases, and only three, viz.: $m > n$, $m < n$, $m = n$.

1°. $m > n$. In this case, $\frac{Nx^m}{Dx^n} = \frac{Nx^{m-n}x^n}{Dx^n}$.

When $x=0$, $\frac{Nx^m}{Dx^n} = \frac{0}{0}$. Now it is obvious that the factor x^n becoming 0, when $x=0$, causes the vanishing of the fraction. Dividing both terms by x^n , we have $\frac{Nx^{m-n}}{D}$. Making

$x=0$, this fraction reduces to $\frac{N \cdot 0}{D} = \frac{0}{D} = 0$. Hence the true value of this fraction, under the hypothesis $x=0$, is 0. This fact we indicate by the expression $\left(\frac{Nx^m}{Dx^n}\right)_{x=0} = 0$.

Take, as an example, the fraction $\frac{(3a^2+b^2)x^2}{(2a+b)x^2}$. Placing $x=0$, we have $\frac{0}{0}$. But x^2 is evidently common to the terms of this fraction. Dividing both terms by x^2 , we have $\frac{(3a^2+b^2)x}{2a+b}$, which reduces to $\frac{(3a^2+b^2) \cdot 0}{2a+b} = \frac{0}{2a+b} = 0$, when $x=0$.

2°. $m < n$. In this case $\frac{Nx^m}{Dx^n} = \frac{Nx^m}{Dx^{n-m}x^m}$, and the fraction vanishes when $x=0$, on account of the presence, in the terms, of the common factor x^m , which reduces to 0, *under the hypothesis* $x=0$. Dividing both terms of the fraction by x^m , we obtain $\frac{N}{Dx^{n-m}}$; which, if $x=0$, reduces to $\frac{N}{D \cdot 0} = \frac{N}{0} = \infty$. Hence, if $m < n$, $\left(\frac{Nx^m}{Dx^n}\right)_{x=0} = \infty$.

3°. $m = n$. In this case we have $\frac{Nx^m}{Dx^m}$, which vanishes when $x=0$, on account of the presence of the factor x^m , which becomes 0 when $x=0$. Dividing both terms by the common factor x^m , we have $\frac{N}{D}$. Hence, $\left(\frac{Nx^m}{Dx^m}\right)_{x=0} = \frac{N}{D}$.

Scholium I.—The whole discussion turns on the *relative* values of m and n ; their *absolute* values do not affect it.

Scholium II.—It is important to note that x may be either a monomial or polynomial; its *form* is of no moment.

CLASS II.

There may be two cases, and no more, viz.:

1°. The fraction is of the form $\frac{N(y^m+z^m)}{D(y+z)}$, $= \frac{0}{0}$ when $y = -z$.

By (97) $z+y$ is a common factor of the terms when m is odd, and it reduces to 0 when $y = -z$.

$\frac{y^m+z^m}{y+z} = y^{m-1} - y^{m-2}z + y^{m-3}z^2 - \dots + z^{m-1}$. If, in this quotient, we make $y = -z$, the terms involving the odd powers of y will become positive; and as those involving the even powers are positive now (and will not change their sign), every term of the quotient will reduce to $+z^{m-1}$, and as there are m terms, the quotient will become $+mz^{m-1}$.

$$\text{Hence, } \left[\frac{N(y^m+z^m)}{D(y+z)} \right]_{y=-z} = \frac{N}{D} m z^{m-1} = \frac{m N z^{m-1}}{D}.$$

2°. The fraction is of the form $\frac{N(y^m-z^m)}{D(y-z)}$, vanishing when $y=z$. By (96), $y-z$ is a common factor of the terms; and this factor, reducing to 0 when $y=z$, causes the fraction to vanish.

$$\frac{y^m-z^m}{y-z} = y^{m-1} + y^{m-2}z + y^{m-3}z^2 + \dots + z^{m-1} \quad (1).$$

$$\text{Hence, } \frac{N(y^m-z^m)}{D(y-z)} = \frac{N}{D} (y^{m-1} + y^{m-2}z + \dots + z^{m-1}).$$

If $y=z$, each term of (1) reduces to y^{m-1} , and, there being m terms, the quotient (1) becomes my^{m-1} , and the fraction reduces to $\frac{N}{D} my^{m-1}$. Hence, $\left(\frac{N y^m - z^m}{D y - z} \right)_{y=z} = m \frac{N y^{m-1}}{D}$.

From the foregoing discussions, we deduce, for the evaluation of any vanishing fraction, the following

RULE.—Divide both terms of the given fraction by the continued product of all the factors that are common to these terms, and that reduce to zero under the particular hypothesis.

Find the value of the fraction, thus transformed, under that supposition which occasioned the vanishing of the original fraction.

420. Corollary.—If a fraction reduces to $\frac{\infty}{\infty}$ under any supposition, it is a vanishing fraction,

since $\frac{\infty}{\infty} = \frac{1}{0} \div \frac{1}{0} = \frac{1}{0} \times \frac{0}{1} = \frac{0}{0}$, and may be treated by the rule (419).

421. It is sometimes necessary to transform the fraction, by multiplying both terms by some quantity, in order to

show the existence of a common factor, the discovery of which would otherwise be difficult, but, of course, care must be taken not to introduce, in this multiplication, a factor that would itself reduce to 0, under the particular hypothesis. The operation here described is a preparation for the application of the rule; not a step in the rule.

422. The different forms of vanishing fractions do not always occur separately; several forms may be combined in a single fraction, which may therefore vanish for particular values of several of the unknown quantities entering it. If, after cancelling all common factors, the fraction still reduces to $\frac{0}{0}$, this form no longer conceals the ratio, but is itself the true ratio, and the expression is a veritable symbol of indetermination. Except in this case, the results of the evaluation of a vanishing fraction must be in one of the forms, $\frac{0}{N}$, $\frac{M}{0}$, or $\frac{M}{N}$, any one of which may, of course, be either positive or negative.

423. The discussion of vanishing fractions has been confined to fractions reducing to the form $\frac{0}{0}$, under a particular supposition, these forming the most common and most important class. It is clear, however, that if expressions reduce to $\frac{\infty}{\infty}$, $0 \cdot \infty$ or $\infty - \infty$ (as each of these is equivalent to $\frac{0}{0}$), they may conceal the true ratio under these forms. They may be transformed so as to reduce to $\frac{0}{0}$, under the particular hypothesis, and then evaluated as shown. Or, we may ascertain whether or not there are present common factors that become infinite or zero under the particular hypothesis, and then strike out such factors, if any exist, before making the supposition. Take, for example, the fraction $\frac{ax+b}{cx+d}$, which becomes $\frac{\infty}{\infty}$ when $x = \infty$. This

fraction may be placed under the form $\frac{x(a + \frac{b}{x})}{x(c + \frac{d}{x})}$, in which

the presence, in the terms, of the common factor x , which becomes ∞ under the particular hypothesis, is manifest.

Dividing both terms by this factor, we obtain, $\frac{a + \frac{b}{x}}{c + \frac{d}{x}}$,

which becomes $\frac{a}{c}$ when $x = \infty$.

Again, taking the expression, $\frac{ax+b}{cx+d}$, we may place it in the form $\frac{1}{cx+d} \div \frac{1}{ax+b}$, which becomes $\frac{0}{0}$ when $x = \infty$, and therefore falls under a case already discussed.

The expression, $(a+bx)\frac{c+x}{x+dx^2} = \infty . 0$ when $x = \infty$; for $a+bx$ becomes ∞ , and $\frac{c+x}{x(1+dx)} = \frac{(\frac{c}{x}+1)x}{(1+dx)x} = \frac{1+\frac{c}{x}}{1+dx}$, which becomes $\frac{1}{\infty} = 0$, when $x = \infty$. The given expression may be

placed under the form $\frac{\frac{c}{x}+1}{1+dx} \div \frac{1}{a+bx}$, which reduces to $\frac{0}{0}$ when $x = \infty$, thus falling into the class discussed. Or, we

may place the given expression in the form, $\frac{(\frac{a}{x}+b)(\frac{c}{x}+1)x^2}{(d+\frac{1}{x})x^2}$,

in which the presence of the factor x^2 , becoming infinite under the particular hypothesis, is evident. Striking out this factor, and making the required supposition, the value of the expression becomes $\frac{b}{d}$. Other cases might be treated similarly. The device of transforming the fraction or given

expression, to render clear the presence of a common factor, is very often necessary, and skill in its employment is very important. One observation it is well to make before leaving this subject: Whenever a supposition is to be made that leads to results of ∞ , 0, etc., we must beware of failing cases, concealed ratios, etc., and never pass over such results without careful inquiry concerning the circumstances that produce them.

SECTION III.

DISCUSSION OF PROBLEMS.

424. The Discussion of a problem is the operation of making every possible supposition upon the arbitrary quantities entering it, and explaining the results thus obtained.

425. Arbitrary Quantities are those to which any value may be assigned at pleasure.

Arbitrary quantities are not unknown, but enter, as the known quantities, the *general statement* of a problem, and by assigning to them particular values we obtain the *special cases* of the problem.

426. An Indeterminate Problem is a problem that admits of an infinite number of solutions.

A problem is always indeterminate when the conditions are less in number than the unknown quantities involved, since the statement will then involve a less number of equations than of unknown quantities.

PROBLEMS.

427. 1. Find a number that, increased by a , shall be equal to b .

Let x represent the number. Then $x+a=b$ (1) $\therefore x=b-a$.
Here we may have two cases, $b>a$ or $a>b$.

1°. $b > a$. In this case the value of x is positive, and the solution gives the number required, without presenting any difficulty.

2°. $a > b$; the value of x is negative. In this case the subtraction of a from b , in the strict arithmetical sense, is impossible, and hence the solution of the problem, understood in an arithmetical sense, is impossible; we cannot find a number that, increased by a second, shall be less than the second number. The value of x will, however, satisfy (1), and therefore is an answer to the problem taken in its algebraic or general sense, and we see at once that $b - a$ added, in the algebraic sense, to a , will give b as the sum.

To ILLUSTRATE, 1°. Let $b = 73$ and $a = 50$. Then $x = b - a = 23$, and we see at once that $50 + 23 = 73$. 2°. Let $b = 73$, and $a = 97$. Then $x = b - a = -24$. In this case the problem is one admitting no solution, if understood arithmetically; for, 97 being greater than 73, no number, added to 97, can give 73 as a sum. But, understood algebraically, the solution of the problem is not impossible, for -24 , added algebraically to 97, gives 73 as the sum, and is therefore a true answer to the problem, satisfying the equation, and hence the conditions, thereof.

2. Find a number, such that if it be multiplied by a and the product be added to b , the sum will equal d increased by c times the number.

Let x represent the number; then, from the conditions, $ax + b = cx + d$ (1) $\therefore x = \frac{d-b}{a-c}$. Now the different values that may be assigned to a , b , c and d , give rise to several different cases.

1°. Suppose $d > b$ and $a > c$; the value of x will then be positive, and will satisfy the conditions of the problem without presenting any difficulty. We naturally expect that if $a > c$, as $ax > cx$, d will be greater than b .

2°. Suppose $d > b$ and $a < c$. The value of x will then be negative, since its terms will have contrary signs. In this

case the solution of the problem, arithmetically considered, is impossible. For, since $a < c$, $ax < cx$, and the less product would require the addition of the greater of the quantities d , b , to give a sum equal to the sum of cx and the other, whereas the less quantity, b , is added thereto, and ax and b being less than cx and d , respectively, their sum is less than the sum of cx and d . Taking the problem in the algebraic sense, however, the solution is not impossible, for this negative value of x satisfies (1), just as well as the positive value. We can see, too, that, x being negative, of ax and cx , that one is algebraically greater which is numerically the less, and we should expect b , less than d , to be added to the algebraically greater (ax), of the quantities ax and cx , if d is to be added to the algebraically less, to make the sums equal.

We may avoid even the apparent or arithmetical absurdity involved in taking this value of x , thus:—As x is negative, substitute $-x$ for $+x$ in (1); whence, $-ax + b = -cx + d$ (2), and $x = \frac{d-b}{c-a}$, a value numerically equal to $\frac{d-b}{a-c}$, but positive. This value satisfies (2). Modifying the enunciation of the problem to suit the new equation, it becomes: “Find a number such that, if it be multiplied by a , and the product be subtracted from b , the difference will equal d , DIMINISHED by c times the number.” The value of x , from (2), satisfies the conditions of this problem, even in the arithmetical sense.

3°. Suppose $d < b$ and $a > c$. The value of x is negative, and the discussion is similar to the last.

4°. Suppose $d < b$ and $a < c$. Though each term of the fractional value of x is negative, their quotient being positive this value of x satisfies the conditions of the problem, even in the arithmetical sense. This we should expect, since, as $a < c$, $ax < cx$, and b , greater than d , ought to be added to the less of these, *i. e.*, ax , d being added to the greater, to make the two sums equal. Though the value of x thus satisfies the equation, d being less than b , and a less than c ,

the subtractions (1°), of b from d , and (2°), of c from a , are impossible, arithmetically considered, (though simply giving negative remainders, algebraically considered). To avoid this difficulty, we may solve (1) thus: $ax+b=cx+d$, whence $cx-ax=b-d \therefore x=\frac{b-d}{c-a}$, a result equal to $\frac{d-b}{a-c}$, but in which the terms, $(b-d$ and $c-a)$, are both positive, and therefore involve no absurdity when arithmetically considered.

5°. Suppose $d=b$; $a < c$. Here $x=\frac{0}{a-c}=\pm 0$. The equation (1) becomes, under this hypothesis, $ax+b=cx+b$ or $ax=cx$, which, a being greater or less than c , cannot be true unless $x=\pm 0$, when this equation reduces to $0=0$.

Only one of the values $+0$ or -0 , can be taken for either hypothesis.

6°. Suppose $d > b$, $a=c$. Here $x=\frac{d-b}{0}=\pm \infty$. Under this hypothesis (1) becomes $ax+b=ax+d$; which, d and b being unequal, cannot be satisfied, unless x is so large that the inequality of d and b would not affect the relative values of the members; so large, in other words that the addition of d or b to ax would make no perceptible difference. Hence, x must be infinite, to satisfy this equation, and will be positive or negative, according to the supposition taken.

The values of x in (5) and (6) are somewhat peculiar, if the problem be understood in its strict arithmetical sense. Still, $x=0$ and $x=\infty$, are true answers to the problem. Taking $x=0$, if we understand 0 in its arithmetical sense, we have x = no quantity, *i. e.*, there is no quantity that will satisfy the conditions of the problem, understood arithmetically, and, referring to the enunciation, we see that this is the case. Taking zero in its algebraic sense, of a quantity less than any assignable quantity, the solution presents no difficulty; for, ax and cx being infinitely small, the addition of either to b would have no effect, *i. e.*, $b\pm 0=b\pm 0$ or $b=b$.

Taking $x = \pm\infty$, the enunciation of the problem also presents an absurdity, for it is apparently impossible to add the same quantity to two unequals and obtain equal results. But this difficulty is removed when we remember that finite quantities, considered with respect to infinity, are as zeroes, and their addition or subtraction has no effect on the infinite quantities, so that, x being infinite, $ax + b = ax + d$.

7°. Suppose $a=c$, $b=d$; whence $x = \frac{0}{0}$. This we interpret as meaning that x may have any value whatever, and, recurring to the equation and the problem, we see that this is the case, for the equation becomes $ax + b = ax + b$, which is identical, and may, therefore, be satisfied by any value whatever, substituted for x . The problem in this case evidently admits of an infinite number of solutions, *i. e.*, is *Indeterminate*. Whenever, then, $\frac{0}{0}$ is a true answer to a problem, that is, when it does not arise from the disappearance of a common factor in the terms (415), it indicates that the problem admits of an infinite number of solutions.

NOTE.—By assigning numerical values to a , b , c , and d , these different cases may be illustrated in a manner entirely analogous to that employed in Problem 1.

428. Indeterminate problems are not all of the kind just exemplified; any problem is indeterminate when its enunciation involves more unknown quantities than conditions, and thence gives rise to an indeterminate equation. Problems of the class exemplified in 2-7°, may be satisfied by any quantities whatever, and this is always the case when the enunciation of the problem, under any supposition, gives rise to an *identical* equation; but problems giving rise to *indeterminate* equations, though admitting of an infinite number of solutions, are not satisfied by any quantities at pleasure.

TO ILLUSTRATE:—3. Find two numbers, such that a times the first, increased by b times the second, may equal c .

Let x and y represent the numbers. Then, from the conditions, $ax + by = c$. This equation may be satisfied by an

infinite number of *sets* of values, but not by substituting any values whatever for the unknown quantities. The problem, then, admits of an infinite number of solutions, i. e., it is *indeterminate*, though it cannot, like 2-7°, be satisfied by any quantities whatever.

4. Find two numbers, such that a times the first, increased by b times the second, shall be equal to c , and a' times the first, increased by b' times the second, shall be equal to c' .

Let x and y represent the numbers.

Then, from the conditions, $ax + by = c$ (1)

$$a'x + b'y = c' \quad (2)$$

Then $aa'x + a'by = a'c$ (3)

$$aa'x + ab'y = ac' \quad (4)$$

$$(\alpha'b - ab')y = a'c - ac' \quad (5)$$

$$\text{Similarly } (ab' - a'b)x = b'c - bc' \quad (6)$$

$$\text{Hence } x = \frac{b'c - bc'}{ab' - a'b} \quad (7), \text{ and } y = \frac{a'c - ac'}{a'b - ab'} \quad (8).$$

Now suppose that $ab' = a'b$, $b'c < bc'$ and $a'c > ac'$. Then $x = \infty$ and $y = \infty$. These values do not satisfy (1) and (2); for, by substituting, for x and for y , ∞ , we have the sum of two infinite quantities equal to a finite quantity. What, then, are the values of x and y ? Examining (5) and (6), we see that, if $ab' = a'b$, $ab' - a'b$ being equal to $a'b - ab'$, equal to zero, these equations become

$$0.y = a'c - ac' \therefore a'c - ac' = 0 \text{ or } a'c = ac'$$

$$0.x = b'c - bc' \therefore b'c - bc' = 0 \text{ or } b'c = bc'$$

Hence, the equality of ab' and $a'b$ involves the equality of $b'c$ to bc' and of ac' to $a'c$, and we might expect that the values found under our supposition would not satisfy (1) and (2), since the supposition was itself absurd.

If now we take, as we should, the supposition $ab' = a'b$, with the suppositions, $a'c = ac'$, and $b'c = bc'$, we find $x = \frac{0}{0}$, and $y = \frac{0}{0}$; x and y are indeterminate, or each has an infinite number of values. To test these results, let us go back

to (1) and (2); (1) may be written $\frac{a}{b}x + y = \frac{c}{b}$ (9); (2) may be written $\frac{a'}{b'}x + y = \frac{c'}{b'}$ (10); $\therefore ab' = a'b$, we have, dividing both equals by $b'b$, $\frac{a}{b} = \frac{a'}{b'}$. Again, since $b'c = bc'$, $\frac{c}{b} = \frac{c'}{b'}$. Hence equations (9) and (10) are the same; i. e., the problem gives rise to one equation, involving two unknown quantities, which equation is indeterminate, may be satisfied by an infinite number of sets of values of x and y , and hence the problem is indeterminate. The values $x = \frac{0}{0}$, $y = \frac{0}{0}$, are, then, the true answers to the problem, under the supposition, indicating its indeterminate character.

It will be seen that the supposition $ab' = a'b$, whence $\frac{a}{b} = \frac{a'}{b'}$, makes the first members of (9) and (10) the same. Hence the equations cannot be true, unless the second members are equal, i. e., unless

$$\frac{c}{b} = \frac{c'}{b'}, \text{ whence } b'c = bc'. \text{ Since } \frac{a'b = ab'}{bc' = b'c}, \frac{a'}{c'} = \frac{a}{c},$$

whence $a'c = ac'$, so that we may show that the equality of ab' and $a'b$ involves the equality of $b'c$ to bc' and of $a'c$ to ac' , in a second way, independent of that already employed. The value $x = \infty$, must, with some value of y , satisfy the equations (1) and (2). From (1), $y = -\frac{ax - c}{b}$, which, x being infinite, gives $y = -\infty$. Similarly, $y = \infty$, gives $x = -\infty$, which values evidently satisfy both equations. Though x and y , then, have each an infinite number of values, for any one value of either there is but one corresponding value of the other.

By supposing $b'c = bc'$, and $a'c = ac'$, $ab' > a'b$, we should find $x = 0$, $y = 0$; values which evidently do not satisfy the given equations, and, by examining (5) and (6), we could deduce the same results as before, in an analogous manner.

The values of x and y , (7) and (8), then, are so related, that, if either term of either fraction becomes 0, both terms of both fractions become 0, and x and y indeterminate.

If a , a' , b , or b' should reduce to 0, one of the equations would be determinate, giving the value of x or y , which, substituted in the other, would give at once the value of the other unknown quantity. Or, the values of x and y might be obtained, by substitution, from (7) and (8).

If a and b both reduce to 0 at the same time, $c=0$, and $x = \frac{0-0}{0-0} = \frac{0}{0}$, and $y = \frac{0}{0}$ also; similarly, if a' and b' both reduce to 0 at the same time. If a and a' both become 0 at the same time, (1) becomes $0x+by=c$, or $by=c$, whence $y = \frac{c}{b}$. (2) becomes $0y=c'$, whence, $y = \frac{c'}{b'}$. As, in order that (1) and (2) may be simultaneous, y must have the same value in the two, $\frac{c}{b}$ must equal $\frac{c'}{b'}$, or $b'c=bc$.

Hence $x = \frac{0}{0}$, and $y = \frac{c(1-\frac{ac'}{a'b'})}{b(1-\frac{ab'}{a'b'})} = \frac{c}{b}$, since, $b'c$ being equal

to bc , $\frac{c'}{c} = \frac{b'}{b}$. Hence x will be indeterminate, and y will equal $\frac{c}{b}$. This is as it should be; for, the coefficients of x being 0, any value of x will satisfy either equation, the two equations reducing to the same form, from what has just been shown.

Similarly, if b and b' both reduce to 0 at the same time, y will be indeterminate, and x will equal $\frac{c}{a} = \frac{c'}{a'}$.

In general, any unknown quantity whose coefficient is 0, is indeterminate.

If $c=c'=0$, and $a'b < ab'$, $x=0$, and $y=0$. If $c=c'=0$, and $a'b=ab'$, $x = \frac{0}{0}$, and $y = \frac{0}{0}$.

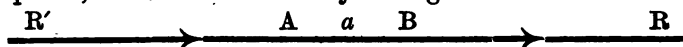
(1) and (2) become

$$ax+by=0 \therefore y=-\frac{a}{b}x; \quad a'x+b'y=0 \therefore y=-\frac{a'}{b'}x.$$

Here we see that, $\frac{a}{b}$ being equal, by supposition, to $\frac{a'}{b'}$, the values of y are equal, no matter what the value of x , and, either x or y being indeterminate, the other is, of necessity, so. If, however, $\frac{a}{b}$ were not equal to $\frac{a'}{b'}$, it is clear that the two values of y could not be the same, unless x were 0, in which case y itself would be zero. In this case, $\frac{y}{x}=-\frac{a}{b}=-\frac{a'}{b'}=\frac{0}{0}$, the equations being untrue unless $x=y=0$; while, if $\frac{a}{b}=\frac{a'}{b'}$, $\frac{y}{x}=-\frac{a}{b}=-\frac{a'}{b'}$, is determinate.

PROBLEM OF THE COURIERS.

429. Two couriers are traveling along the same right line, in the same direction. One travels m miles per hour, and the other n miles; at 12 m. they are a miles apart; required, the time at which they are together.



Let $R'R$ be the line along which they travel, A and B being the respective positions at 12 m., the arrows indicating the direction in which they travel, and R' or R the point at which they are together.

Suppose all distances to be measured from A ; positive distances to the right, and therefore negative distances to the left. The distance a , then, as represented on $R'R$, being measured from A toward B , will be positive. If B were to the left of A , this distance would be negative.

Let t = the number of hours from 12 to the time at which the couriers are together.

Let x = the number of miles that the B courier travels in t hours, at the rate of n miles per hour.

Then will $a+x$ = the number of miles that the other travels in t hours, at the rate of m miles per hour.

Since the former travels t hours, at the rate of n miles per hour, he travels altogether nt miles, which, of course, must equal x , the number of miles that he travels in t hours, or, $nt=x$ (1). Similarly, $mt=a+x$ (2). Solving (1) and (2), we find, $t=\frac{a}{m-n}$; $x=\frac{na}{m-n}$; $a+x=\frac{ma-na+na}{m-n}=\frac{ma}{m-n}$.

From the nature of the case, neither m nor n can be negative; our suppositions, then, must be principally on their *relative* values.

There may be six cases, viz.: $a>0$ $\begin{cases} m>n \\ m=n, \text{ and } a=0 \\ m<n \end{cases}$

I. $a>0$.

1°. $m>n$. Here $m-n$ is positive, and, a being positive, $t=\frac{a}{m-n}$ is positive. We interpret this result as indicating that the couriers are together *after* 12 o'clock.

This is what we should expect from the conditions of the problem, for, m being greater than n , the courier from A travels faster than the one from B. He is therefore constantly gaining on the latter, and must eventually overtake him, and must overtake him *after* 12 m.

$x=\frac{an}{m-n}$ and $a+x=\frac{am}{m-n}$ being both positive, we conclude that the couriers will be together to the *right* of A, since all positive distances are measured to the right, from this point. This is in accordance with the conditions, for the couriers at 12 m. leave the places A and B, and travel toward the right. Hence, at any time after 12 m. they will be found to the right of A.

t in this case is the number of hours to be *added* to 12, to find the time at which they are together.

2°. $m<n$. In this case t is negative, and we interpret this as meaning precisely the opposite of the positive result; i. e., we interpret it as indicating that the time at which they are together is *previous* to 12 m. In this case, t is the number of hours to be *subtracted* from 12, to obtain the time at which they are together.

Our interpretation is in accordance with the conditions of the problem, for, m being less than n , the forward courier travels faster than the courier from A. He will therefore be continually getting farther and farther from the A courier, and, the distance between them constantly increasing, they will never be together after 12 M. As the interval between them is continually increasing, and is a miles at 12 M. (and the rate of travel does not change), it must have been less than a miles before 12, and the longer the time before 12 the shorter must have been the interval, until at some time before 12 this interval must have been 0, i. e., at some time before 12 the couriers must have been together. The exact hour we find by subtracting t hours from 12. Hence, at $(12-t)$ o'clock A. M. the couriers were together. Before this time the one who is now in advance was behind the other, but continually gaining upon him. He passed him at this hour, and will continue to separate himself from him as long as they travel. As the interval is always measured from the A courier, when he is in advance it is measured to the left, or in a negative direction. In this case $\frac{a}{m-n}$ will be positive when $m < n$, and negative when $m > n$, and the time until the couriers are together will be, in the one case, a number to be added to the number of hours, or the time o'clock at which they are a miles apart, and in the other case to be subtracted, and the discussion will not differ essentially from that just given.

When $m < n$, $x = \frac{na}{m-n}$, and $a+x = \frac{ma}{m-n}$, are both negative, showing that the couriers are together at some point R', to the left of A, as would be supposed from the conditions of the problem.

It will be seen that if the value of t in 1° be considered the answer to a problem stated as before, with the modification, "at what time after 12 M. will they be together?" the

value of t in 2° will be an answer to the same problem with the modification, "at what time *before* 12 m. were they together?"

3°. $m=n$. Then $t=\frac{a}{0}=\infty$. This result we interpret as indicating that the couriers will never be together, and have never been together.

This interpretation is in accordance with the conditions of the problem; for, if $m=n$, the two couriers travel at the same rate, and therefore the interval between them is always the same. As this interval is >0 at 12 m., it has always been, and always will be, >0 , and equal to a ; i. e., the couriers have never been, and will never be, together, but will always be separated by an interval of a miles, just as they are at 12 m.

In this case, $x=\frac{na}{m-n}=\frac{na}{0}=\infty$, and $a+x=\frac{ma}{0}=\infty$, i. e., the point at which the couriers will be together is at an infinite distance from A. If the B courier is in advance, this distance will be measured to the right of A. By changing its sign, we find the distance, $-\infty$, to the left, at which the couriers *were* together, taking the problem in the second sense, indicated in 2°.

II. $a=0$.

1°. $m > n$. Then $t=\frac{0}{m-n}=\pm 0$. We interpret this result as indicating that the couriers are together at 12 m. The value $+0$, indicates that the A courier has just overtaken the B courier. The value -0 , indicates that the latter has just overtaken the former. These results are in accordance with the conditions; for, if the A courier travels the faster, he will be continually gaining on the other; and as the two are together at 12 m., he must have just overtaken the B courier, and after 12 m. will be continually separating himself from his noonday companion. If the latter travels the faster, of course the reverse takes place.

In this case, $x = a + x = \frac{na}{m-n} = \frac{ma}{m-n} = 0$; which results indicate that the couriers are together precisely at A, as we should naturally expect from the conditions.

2°. $m=n$. Then $t = \frac{0}{0}$. This, the symbol of indetermination, we interpret as indicating that t may have any value whatever, i. e., that the couriers are together at any time, before or after 12 m.; in other words, that they are always together.

This interpretation is in accordance with the conditions; for, since the couriers are together at 12 m., and travel at the same rate, they have always been, and will always be, together.

Under this hypothesis, x and $a+x$ both reduce to $\frac{0}{0}$; indicating that at any point whatever on the line R'R, the couriers are together, which accords with the given conditions.

430. *Scholium.* $t = \frac{a}{m-n}$, $x = \frac{na}{m-n}$, and $a+x = \frac{ma}{m-n}$ are general formulæ, which may be used in solving any "problems of pursuit," of which class the one under discussion may be considered a general representative.

431. From the foregoing discussions we deduce the following

GENERAL RESULTS.

The values found for the unknown quantities involved in any problem may take any one of *five general forms*.

1°. Positive values.

These are, in general, answers to the problems in the strict sense in which they are proposed. There are some cases, however, e. g., Prob. 31, p. 250, in which even a positive result, if fractional, would not satisfy the conditions of the problem. If a positive value, found for the unknown quantity, does not satisfy all the conditions of a problem, the solution of the problem (in the strict sense proposed), is impossible.

2°. Negative results.

(a.) *Negative results indicate the impossibility of solving the problems given, if these problems are taken in the strict arithmetical sense.*

(b.) *Any such negative result, taken with a contrary sign, may be considered the answer to a problem of which the enunciation differs from the one proposed, only in that quantities that were additive have become subtractive, and the reverse.*

(c.) *If the problem be taken in a general sense, the negative value of the unknown quantity is a true answer, but must be interpreted in a sense contrary to the sense in which a positive value is interpreted.*

3°. Results of the form $\frac{0}{N}$, or 0.

A result of this form may indicate the impossibility of solving the problem, in its strict arithmetical sense, or may be a true answer to the problem, taken in the algebraic sense.

4°. Results of the form $\frac{M}{0}$, or ∞ .

A result of this form may, in general, be considered as indicating the impossibility of satisfying the conditions of the problem, in finite quantities.

5°. Results of the form $\frac{0}{0}$.

A result of this form that does not arise from the vanishing of a fraction on account of the presence of a factor common to its terms, indicates that the given problem admits of an infinite number of solutions.

PROBLEMS GIVING RISE TO EQUATIONS OF THE SECOND DEGREE.**PROBLEM OF THE LIGHTS.**

432.—1. Find, on the line joining two lights of given intensities, a point equally illuminated by the lights.

In solving this problem, we employ the following principle of optics, viz.:

The intensity of a light at any distance is equal to the in-

intensity of the light at the distance 1, divided by the square of the distance.

P'' L P L' P'

Let P''P' be the indefinite line joining the lights, L and L'. Assume L as the origin of distances, all distances measured to the right of L being positive, and hence all measured to the left being negative. Let P be the point required, and let d represent the distance, L L', between the lights. Let x = the number of units (feet, metres, etc.) from L to the point of equal illumination, P. Then $d-x$ represents the distance from L' to P. Suppose the intensity of the light L, at the distance 1, to be a . From the principle of optics already referred to, the intensity of this light at the distance $2 = \frac{a}{4}$, at the distance $6 = \frac{a}{36}$, etc.,

and at the distance x , i. e., at the point P, is $\frac{a}{x^2}$. Suppose the intensity of L', at the distance 1, is b ; then at the distance $d-x$, i. e., at the point P, the intensity of this light is $\frac{b}{(d-x)^2}$. From the conditions of the problem, the intensities of the two lights being equal at P, $\frac{a}{x^2} = \frac{b}{(d-x)^2}$,

whence, $\frac{(d-x)^2}{x^2} = \frac{b}{a}$; $\frac{d-x}{x} = \pm \frac{\sqrt{b}}{\sqrt{a}}$; $\pm \sqrt{b}.x = \sqrt{a}.d - \sqrt{a}.x$;

$(\sqrt{a} \pm \sqrt{b}).x = d\sqrt{a} \therefore x = \frac{d\sqrt{a}}{\sqrt{a} \pm \sqrt{b}}$ (1); $x' = \frac{d\sqrt{a}}{\sqrt{a} - \sqrt{b}}$ (2);

$d-x = \frac{d\sqrt{b}}{\sqrt{a} + \sqrt{b}}$ (3); $d-x' = -\frac{d\sqrt{b}}{\sqrt{a} - \sqrt{b}}$.

From the nature of the case, neither a nor b can be negative, whence we see that the values of x must always be real. There are, therefore, always two points of equal illumination on P''P'. We naturally suppose that there would be such a point, not only between L and L', but also beyond them, on the side of the less intense light.

Moreover, as d is measured from L , toward the right, it can never be negative.

1°. Suppose $d > 0$; $a > b$. Both values of x are, in this case, positive; both must, then, be measured toward the right; take the first value. Since,

$$\sqrt{a} + \sqrt{b} > \sqrt{a}, \quad \frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}} < 1, \text{ and hence } x = \frac{d\sqrt{a}}{\sqrt{a} + \sqrt{b}} < d;$$

i. e., the distance from L to the first point is less than the distance from L to L' . We interpret this as indicating that the first point of equal illumination lies between L and L' . Again, as $\sqrt{a} + \sqrt{b} < 2\sqrt{a}$ ($\because \sqrt{a} > \sqrt{b}$), the numerator being greater than $\frac{1}{2}$ the denominator, the fraction

$$\frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}} > \frac{1}{2} \text{ and } x = \frac{d\sqrt{a}}{\sqrt{a} + \sqrt{b}} > \frac{d}{2}.$$

We interpret this as indicating that the point sought is to the right of the middle point of d , i. e., is nearer to L' than to L . This is in accordance with the conditions; for, L being the more intense light, the point of equal illumination is necessarily farther from it than from the light L' ; that is, the point, between them, that is equally illuminated, must be nearer the less intense light.

The corresponding value of $d - x$, $\frac{d\sqrt{b}}{\sqrt{a} + \sqrt{b}}$, is positive,

and, since $\sqrt{a} + \sqrt{b} > 2\sqrt{b}$, is less than $\frac{d}{2}$, i. e., the distance, from L' , to the point of equal illumination, is less than half the distance to L .

The second value of x , $\frac{d\sqrt{a}}{\sqrt{a} - \sqrt{b}}$, $> d$, since $\sqrt{a} > \sqrt{a} - \sqrt{b}$,

and therefore, $\frac{\sqrt{a}}{\sqrt{a} - \sqrt{b}} > 1$. We interpret this as indicating that the second point of equal illumination is beyond and to the right of L' , as at P' , which interpretation is evidently in accordance with the conditions of the problem.

The corresponding value of $d-x$ is negative. The first value, being positive, was estimated to the right, from P toward L', and this one should therefore be negative, being estimated from P' toward the left.

2°. Suppose $d > 0$; $a < b$. The first value of x , being positive, is to be estimated from L toward the right, and since $\sqrt{a} + \sqrt{b} > 2\sqrt{a}$, $\frac{\sqrt{a}}{\sqrt{a} + \sqrt{b}}$ is a fraction less than $\frac{1}{2}$, whence $x < \frac{d}{2}$. We interpret this as indicating that the first point

equally illuminated is between L and L', and at a distance from L less than half the distance to L'. This is in accordance with the conditions, for the point equally illuminated between the lights should be nearer to the less intense light.

The second value of x is negative, since $d\sqrt{a}$ is positive and $\sqrt{a} - \sqrt{b}$ is negative, as a is less than b . Hence this value must be estimated to the left of L, as to P''. This is as it should be, for the point, beyond the lights, that is equally illuminated, should be on the side of, and nearer to, the less intense light.

3°. Suppose $d > 0$; $a = b$. Then $x' = \frac{d\sqrt{a}}{2\sqrt{a}} = \frac{d}{2}$; which result we interpret as indicating that the first point equally illuminated is to the right of L, at a distance equal to half that separating the lights. This is manifestly as it should be; for, the lights being equally intense, the point midway between them is the only point between them that can be equally illuminated by them.

$x' = \frac{d\sqrt{a}}{\pm 0} = \pm \infty$. We interpret this result as indicating that there is no point beyond the lights, at a finite distance therefrom, equally illuminated by them. This is in accordance with the conditions; for, a being equal to b , there can be no point beyond either L or L' at which the intensity of the light considered equals the intensity of the other, except at an infinite distance, when the intensity of each light be-

comes a finite quantity divided by infinity, that is, becomes 0, and the point is in total darkness.

4°. Suppose $d=0$; $a>b$ or $a<b$. Then $x' = \frac{0\sqrt{a}}{\sqrt{a}+\sqrt{b}} = 0$,

and $x'' = \frac{0\sqrt{a}}{\sqrt{a}-\sqrt{b}} = 0$. We interpret these as indicating that both points of equal illumination coincide at L. This is in accordance with the conditions; for, since the lights are unequal in intensity, and are situated at the same point, there can be no point of P''P' on either side of L at which the intensities of the lights would be equal. If there could be such a point at any finite distance, as x , from L, the intensities of the lights at this point would be $\frac{a}{x^2}$ and $\frac{b}{x^2}$ respectively, which can never be equal unless $a=b$, and this is contrary to the hypothesis. Again, the intensity of either light at the distance 0 is a finite quantity divided by 0, i. e., is infinity, so that the intensities of the two lights at the point at which they are situated are equal, each to ∞ .

5. Suppose $d=0$; $b=a$. Then $x' = \frac{0\sqrt{a}}{\sqrt{a}+\sqrt{b}} = 0$, and

$x'' = \frac{0\sqrt{a}}{0} = 0$. The value of x'' is indeterminate; for, re-

curring to the equation $\frac{a}{x^2} = \frac{b}{(d-x)^2}$, or $a(d-x)^2 = bx^2$, or $(a-b)x^2 - 2adx = -d^2a$, it becomes under this hypothesis $0 \cdot x^2 - 0 \cdot x = -a^2 = 0$, which is evidently satisfied by any value of x that may be substituted therein. We interpret these results as indicating that every point of P''P' is equally illuminated by L and L', the value $x'=0$ being one of the infinite number of values that may be assigned to x . This interpretation is obviously in accordance with the conditions of the problem; the lights, equal in intensity, coincide at L, which point, as well as every other in the line, must be equally illuminated by them.

The suppositions $a=0$, $b=0$, are not of sufficient importance to justify their discussion.

2. Find a number, such that a times its square, increased by b times the number, shall equal c .

Let $x =$ the number. Then, from the conditions,

$$ax^2 + bx = c \quad (1); \text{ whence } x = \frac{-b \pm \sqrt{b^2 + 4ac}}{2a}.$$

If, now, we suppose $a=0$, $x = \frac{0}{0}$, or $-\frac{2b}{0} = \pm\infty$, giving ∞ the double sign, as zero may be considered the limit of a numerically decreasing quantity, negative or positive. When $a=0$ (1) becomes, $bx=c$, whence $x = \frac{c}{b}$, a finite and determinate value. What, then, is the meaning of the value, $x = \frac{0}{0}$? The value of x is, in this case, a vanishing fraction, and as the vanishing factor is not obvious as the fraction stands, let us multiply both terms by

$$-b - \sqrt{b^2 + 4ac}, \text{ thus obtaining } \frac{-4ac}{-2a(b + \sqrt{b^2 + 4ac})},$$

in which the presence of the common factor $2a$, which reduces to 0 when $a=0$, is evident. Evaluating the fraction, we obtain $x = \frac{c}{b}$, the value found before.

The value $x = \pm\infty$, may be interpreted as indicating that the problem, under the supposition $a=0$, admits of but one solution in finite numbers, which is evidently the case, as the problem then gives rise to an equation of the first degree. This result indicates that, as a becomes smaller and smaller, the second value of x increases numerically without limit. Though the equation (1) has apparently three roots, viz.: $x = \frac{c}{b}$, $x = +\infty$, $x = -\infty$, it has in reality but two, as, in supposing a to decrease to 0, we must suppose that it is *either* positive or negative; we cannot take *both* suppositions at once.

It is well to observe that, when $a=0$, the coefficients, p and q , in the general form (378), become infinite; we must, therefore (as always in approaching 0 or ∞), be on our guard against failing cases.

If $a=0$, $b=0$, $c>0$, both values of x reduce to $\frac{0}{0}$, and, evaluating, we find $x=\pm\infty$, which indicates that the problem under this supposition admits of no solution in finite numbers, as is evidently the case.

Finally, if $a=0$, $b=0$, $c=0$, the values of x become really indeterminate, as they should, since, under this hypothesis, any number whatever will satisfy the conditions of the problem.

CHAPTER IX.

INEQUATIONS.

433. An Inequality, or Inequation, is an algebraic expression of the inequality of two quantities; as $a > b$, $m + n > 2 - c$.

In discussing inequalities, the words *greater* and *less* must be understood in their *algebraic* sense. For example, a negative quantity is less than zero, for we shall evidently obtain a greater result by subtracting zero from any quantity than by subtracting any positive quantity; i. e., $0 > -a$. Again, of two negative quantities, that is the greater which contains the fewer units. Thus, -3 is greater than -7 , for -3 means that 3 is to be subtracted from some quantity, and -7 means that 7 is to be subtracted from some quantity, and we shall evidently obtain a larger result on subtracting 3 from any quantity, than on subtracting 7 from that quantity.

434. Two inequations subsist in the same sense when the greater quantity is in the same member of both; i. e., when the sign of inequality has the same position in both.

ILLUSTRATION. $3 < 7$, $5 < 12$, $2 < 26$ subsist in the same sense; so, also, $5 > 1$, $7 > \frac{2}{3}$, $25 > \sqrt{3}$.

435. Two inequations subsist in a contrary sense when the greater quantity is in the first member of one and in the second member of the other.

ILLUSTRATION. Thus, $2 < 7$ and $12 > 3$ subsist in a contrary sense; so, also, $5 < 27$ and $\sqrt[10]{7} > \sqrt[10]{3}$; $2 < 12$ and $ab > c$; $3 < 11$ and $17 > 9$.

Subsisting in the same sense does not require the difference of the members to be the same in the two inequations,

nor does subsisting in a contrary sense imply that these differences should be unequal.

436. Inequations are classified in the same manner as *Equations*, and, with the exceptions to be noted hereafter, Inequations are subjected to the same transformations and operations as *Equations*.

437. To Solve an Inequation is to transform it so that the unknown quantity may stand alone in one member, and a known quantity alone in the other. This known quantity is called a **Limit** of the unknown quantity.

438. *If the same quantity be added to, or subtracted from, both members of an inequation, the resulting inequation will subsist in the same sense as the original.*

Suppose $a > b$. Then $a - b$ is positive; let it be represented by d ; whence, $a - b = d$, or $a = b + d$ (1); taking any quantity, as x , $a \pm x = b + d \pm x$ (2). If now we subtract d from the second member of (2), and not from the first, we shall evidently leave the second less than the first, i. e., $a \pm x > b \pm x$.

439. Corollary.—*The rule (350) for transposition applies, without change, to inequations.*

Suppose $ax + b > cx + d$ (1). Let it be required to transpose cx to the first member, and b to the second. Subtracting $cx + b$ from each member of (1), we obtain $ax - cx > d - b$, which transformation might evidently have been effected at once by (350).

440. *Multiplying or dividing both members of an inequation by the same positive quantity does not change the sense of the inequation.*

This proposition enables us to clear an inequation of fractions.

441. *Multiplying or dividing both members of an inequation by the same negative quantity, results in an inequation subsisting in a contrary sense.*

This is a direct consequence of (440), for if $a > b$, (1), $ma > mb$, m being entire or fractional, but positive. Let $ma - mb$, which is positive, be d ; then $ma = mb + d$. Transposing, $-mb = -ma + d$. Subtracting d from the second member only, $-mb > -ma$; or, $-ma < -mb$. Hence, multiplying each number of (1), by $-m$ (entire or fractional), would result in an inequation subsisting in a contrary sense.

442. Corollary.—*If the signs of the two members of an inequation be changed, the sense of the inequation is changed*

443. *Both members of an inequation being raised to the same power: If the members are both positive, the resulting inequation subsists in the same sense. If both are negative the resulting inequation subsists in the same sense, if the exponent of the power be odd; in a contrary sense if it be even.*

If $a > b$, a^m is greater than b^m , for it is equal to b^m + other quantities. Then $-b > -a$ (442), and $\therefore a^m > b^m$, $(-b)^m > (-a)^m$, if m be odd. If it be even, $(-b)^m = b^m$ and $(-a)^m = a^m$. Hence, if $-b > -a$, $a^m > b^m$.

444. *If the members of an inequation have contrary signs, they may be raised to any power of an odd degree without destroying the sense. If the power to which they are raised is of an even degree, the resulting inequality will subsist in the same or a contrary sense, according as the positive or the negative quantity is numerically the greater.*

445. *Extracting like roots of the two members of an inequation does not change the sense, if, when the index is even, only the positive roots be compared.*

When the index is even the two members must be positive, as imaginary quantities cannot be compared.

446. *The addition member to member, or multiplication member by member, of two inequations subsisting in the same sense, results in an inequation subsisting in the same sense.*

Suppose $a > b$; $a = b + d$ (1)
 $c > m$; $c = m + d'$ (2) Adding these equations,
 member to member, $a + c = b + m + d + d'$ (3), and subtract-
 ing $d + d'$ from the second member of (1), $a + c > b + m$.
 Again, multiplying (1) by (2), member by member,
 $ac = bm + bd' + md + dd'$, whence $ac > bm$.

This may be extended to three or more inequations.

447. Corollary.—*The subtraction of an inequation from another subsisting in the same sense does not necessarily result in an inequation subsisting in the same sense.*

Subtracting equation (2) (446) from (1), we have,
 $a - c = b - m + d - d'$ (4). Now if $d > d'$, $d - d'$ is positive,
 and its subtraction from the second member of (4) will make
 this member less than the first. If, on the other hand,
 $d < d'$, $d - d'$ is negative, and since the subtraction of a neg-
 ative quantity is equivalent to the addition of a numerically
 equal positive quantity, the subtraction of $d - d'$ from the
 second member of (4) will make this member greater than
 the first. Hence, the resulting inequation may take either
 of the forms: $a - c > b - m$ or $a - c < b - m$.

448. The rules for Elimination may be applied to in-
 equations (noting the exceptional cases already discussed),
 thus frequently enabling us to solve simultaneous inequa-
 tions.

EXAMPLES.

$$1. \frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{6} + \frac{x}{12} > 16. \quad \text{Ans. } x > 12.$$

$$2. x^2 > \frac{px^2}{x-1}. \quad \text{Ans. } x > p + 1.$$

$$3. \begin{cases} \frac{ax}{5} + b(x-a) > \frac{a^2}{5} \\ \frac{bx}{7} - a(x-b) < \frac{b^2}{7} \end{cases} \quad \text{Ans. } x > a \text{ and } x < b.$$

$$4. \frac{a^2 - b^2}{6d} > \frac{a^2 - b^2}{3x}. \quad \text{Ans. } x > 2d.$$

$$5. 7x - 35 + \frac{ax}{6} > 22 + \frac{a}{12}; 8x + 12 - \frac{x}{3} < 70 + a.$$

CHAPTER X.

SERIES.

PRELIMINARY DEFINITIONS.

449. A Series is a succession of terms, each one of which, after the first, or several of the first, is derived from one or more of the preceding, in accordance with some fixed law. This law is called the **Law of the Series**.

ILLUSTRATION. $x^m + x^{m-1}y + x^{m-2}y^2 + \dots + y^m$ is a series; so, also, the second member of the binomial formula.

450. CLASSIFICATION.—Series are:

1°. *Infinite* or *Finite*.

An Infinite Series is one in which there is an infinite number of terms; *e. g.*, the second member of the binomial formula, when m is fractional or negative.

A Finite Series is one in which the number of terms is finite; *e. g.*, the second member of the binomial formula, when m is integral and positive.

2°. *Increasing* or *Decreasing*.

An Increasing Series is one in which every term is numerically greater than the preceding term; *e. g.*,

$$\div 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \dots$$

A Decreasing Series is one in which each term is numerically less than the term immediately preceding; *e. g.*,

$$\div 1 : \frac{1}{2} : \frac{1}{4} : \frac{1}{8} : \dots$$

The terms *Ascending* and *Descending* are used interchangeably with *Increasing* and *Decreasing*, respectively.

3°. *Converging*, *Diverging*, or *Undetermined*.

A Converging Series is a series in which the terms decrease so rapidly toward the right that the sum of an infinite number of terms is a finite quantity.

Series converge *rapidly* or *slowly*, according as the terms fall off, in value, rapidly or slowly; the more rapidly a series converges, the smaller will be the error committed in taking the sum of a finite number of terms for the sum of the whole series. Series may converge from the first term onward, or from any other term onward. It will be seen at once, that converging series are the most useful and important series.

A Diverging Series is a series, the sum of all the terms of which is infinite.

An Undetermined Series is one, of which the sum of all the terms, though finite, is undetermined.

4°. Series are also classified with regard to the nature of the *Law of the Series*.

The most important series, in this connection, are the following:

An Arithmetical Progression, or Progression by Differences, is a series every term of which is derived from the preceding term by the addition of a fixed quantity.

This fixed quantity is called the **Common Difference**, and is positive in an increasing progression and negative in a decreasing one; it may always be obtained by subtracting any term of the progression from the succeeding term.

ILLUSTRATION. $a \dots a \pm \delta \dots a \pm 2\delta \dots a \pm 3\delta \dots$ is an arithmetical progression, the common difference in which is $\pm \delta$.

A Harmonical Progression is a series, the terms of which are so related that their reciprocals form an arithmetical progression.

ILLUSTRATION. $\frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}, \frac{1}{13}, \dots$ is a harmonical progression.

A Geometrical Progression, or Progression by Quotients, is a series, every term of which may be obtained by multiplying the preceding term by a fixed quantity.

This fixed multiplier is called the **Ratio or Rate** of the progression, being the ratio of each term to the next one; in an increasing progression the ratio >1 , and in a decreasing progression <1 ; it may always be obtained by dividing any term by the preceding one.

When a finite number of terms of an arithmetical, harmonic, or geometrical progression is taken, the first and last considered are called, respectively, the **first and last terms of the progression**, and together are called the **extremes**; the intermediate terms are called the **means**, **arithmetical**, **harmonical**, or **geometrical**, as the case may be.

A **Recurring Series** is a series in which any term after the first, or several of the first, is the algebraic sum of the products obtained by multiplying one or more of the preceding terms by one or more fixed quantities.

These fixed quantities, taken in their order, form the **Scale of Relation of the Series**, or, simply, the **Scale of the Series**; the coefficients of the terms in the scale form the **Scale of the Coefficients**.

The **Order** of a Recurring Series is denoted by the number of terms in the scale; a series with only one term in the scale is of the *first* order, one with two terms, of the *second* order, and so on.

ILLUSTRATION. $1+2x+3x^2+4x^3+5x^4+\dots$ is a recurring series of the second order; each term after the second is the sum of the products of the preceding and second preceding terms, into $2x$ and $-x^2$, respectively.

The **Binomial Formula** has been already discussed, and other series will be discussed hereafter.

A series is often named by the expression from which it is derived, as the **Logarithmic series**.

The **First Order of Differences** of a series is the succession of terms obtained by subtracting each term of the series from the succeeding term. The **Second Order of Differences** is obtained from the first, as the first is obtained from the original series. The **Third Order of Differences** is obtained, in like manner, from the second, and so on.

5°. Series are classified according to the nature of the successive *Orders of Differences*.

A series having all the terms of the **First** order of differences equal, is called an **Arithmetical Progression of the First Order**, or, simply, an **Arithmetical Progression**; a series from which the **Second** order of differences has all the terms equal, is called an **Arithmetical Progression of the Second Order**; and, in general, a series having all the terms of its *n*th order of differences equal, is an **Arithmetical Progression of the *n*th Order**. The term *Arithmetical Progression*, alone, always signifies a progression of the first order, as already defined. The progressions of higher orders than the first are called, collectively, **Higher Progressions**.

451. Interpolation is the operation of inserting, between the terms, two and two, of a series, new terms, conforming to the general law of the series.

452. The Summation of a series is the process of finding an expression for the sum of any number of terms of the series.

453. Ratio is the relation of one quantity to another of the same kind.

Ratio may be *Arithmetical*, *Harmonic*, or *Geometrical*.

The Arithmetical Ratio of one quantity to another is the remainder obtained by subtracting either from the other.

ILLUSTRATION. The arithmetical ratio of *a* to *b* is $a-b$, or $b-a$.

The Harmonic Ratio of one quantity to another is the arithmetical ratio of the reciprocals of the given quantities.

ILLUSTRATION. The harmonic ratio of *a* to *b* is $\frac{1}{a} - \frac{1}{b}$, or $\frac{1}{b} - \frac{1}{a}$.

The Geometrical Ratio of one quantity to another is the quotient arising from the division of either by the other.

ILLUSTRATION. The geometrical ratio of *a* to *b* is $\frac{a}{b}$ or $\frac{b}{a}$.

Writers in these states generally prefer the second method of taking the above ratio. As the sign ($:$) of ratio is, however, simply one form

of the sign of division, being used by many European writers to the exclusion of all other signs of division, the ratio $a:b$ is, strictly speaking, $\frac{a}{b}$, and $b:a$ is $\frac{b}{a}$, the ratio of a to b being expressed in either way. It is a matter of little consequence which method be employed, provided only that all ratios be treated in the same way.

The ratio $a:b = \frac{a}{b}$ is often called the *Direct*, and $b:a = \frac{b}{a}$ the *Inverse* or *Reciprocal*, ratio of a to b . It will be seen that the inverse ratio of a to b is the direct ratio of the reciprocals of a and b ; i. e., $\frac{b}{a} = \frac{1}{a} \div \frac{1}{b}$. A ratio is always written, however, as a direct ratio.

The first term of a ratio is called the *antecedent*, and the second term the *consequent*.

The ratio of the *squares* of two quantities is sometimes called the *duplicate*, and the ratio of the *cubes* the *triplicate* ratio of the quantities, while the ratios of the *square* and *cube roots* are called, respectively, the *subduplicate*, and *subtriplicate*, ratios of the quantities; but this mode of expression is undesirable, and is, fortunately, passing out of use.

454. A Proportion is an algebraic expression of the equality of ratios of the same kind, and is called an **Arithmetical**, a **Harmonic**, or a **Geometrical** proportion, according as the ratios compared are *Arithmetical*, *Harmonic*, or *Geometrical*.

If but two ratios are compared, the expression of their equality is called a *Proportion*, simply; if more than two are compared, this expression is called a *Continued Proportion*.

The fourth term of a proportion is said to be a *fourth proportional* to the other three, or, if the second and third are equal, a *third proportional* to the other two. In this latter case, either the second or third is said to be a *mean proportional* or *mean* (Arithmetical, Harmonic, or Geometrical,) between the other two. In general, the *first* and *fourth* terms of a proportion are called the *extremes*, and the *second* and *third* the *means*, the *antecedents* and *consequents* of the ratios preserving their names in the proportion. The divisions made in the proportion by the sign $=$ or $:$ are called *couplets*, and are numbered, *first*, *second*, etc.

The progressions, Arithmetical, Harmonic, and Geometrical, are continued proportions, (arithmetical, harmonic, and geometrical, respectively,) in which any term is a mean between the preceding and succeeding terms.

When the words *ratio* and *proportion* are used without qualification, *geometrical ratio* and *proportion* are to be understood.

Quantities are in proportion by **alternation** when antecedent is compared with antecedent, and consequent with consequent.

Quantities are in proportion by **inversion** when antecedents are made consequents, and consequents are made antecedents.

Quantities are in proportion by **composition** when the *sum* of antecedent and consequent is compared with either antecedent or consequent, and in proportion by **division** when the *difference* of antecedent and consequent is similarly compared.

It is important to remember (Davies' Legendre, Bk. II.) that:

455. *If four quantities are in proportion the product of the means will equal the product of the extremes, and conversely.*

If a couplet is common to two or more proportions, the remaining couplets will form a proportion.

If quantities are in proportion, they will be in proportion by alternation, by inversion, by composition, and by division.

If quantities are in proportion, any equimultiples of one couplet will be proportional to any equimultiples of any other couplet.

If two proportions be multiplied together, term by term, the products will be proportional.

456. Two quantities are proportional to each other when the ratio between them remains the same, whatever changes in value the quantities themselves may undergo. Either quantity is said to be *directly proportional* to the other.

Thus, xy (x varies as, or is proportional to, y) when $\frac{y}{x}$ remains the same, however x and y may change. The proportion, xy , is an example of **General Proportion, or Variation**.

Two quantities are *reciprocally*, or *inversely proportional*, when either is proportional to the reciprocal of the other. Thus, x is inversely proportional to y when $x \propto \frac{1}{y}$. This is an example of **Inverse Variation**.

Before proceeding in the treatment of the problems met with in the discussion of series we shall examine some of the methods in which series are obtained.

SECTION I.

DEVELOPMENT OF EXPRESSIONS INTO SERIES.

457. To Develop an algebraic expression into a series is to transform it into an equivalent series.

The methods of developing an expression into a series are four in number, viz.:

- I. THE METHOD OF DIVISION.
- II. THE METHOD OF MULTIPLICATION, OR INVOLUTION.
- III. THE METHOD OF EVOLUTION.
- IV. THE METHOD OF INDETERMINATES' COEFFICIENTS.

458. I. Method of Division.—A fraction may always be developed into a series by dividing the numerator by the denominator by (89).

This cannot alter the value of the fraction, as it consists simply in performing an indicated operation. The series will, of necessity, be arranged according to the powers of the leading letter of the dividend and divisor. To illustrate:

Let it be required to develop $\frac{a}{a' + x}$ into a series arranged ac-

cording to the ascending powers of x . Dividing, we obtain a series in which each term may be obtained by multiplying the preceding term by $-\frac{1}{a}x$, i. e., we have a geometrical progression, of which the ratio is $-\frac{1}{a}x$. As the division is inexact, we can never reach the last term of the quotient, i. e., the series obtained is infinite. In a similar way we may develop any fraction into a series.

EXAMPLES.

1. Develop $\frac{x^m+y^m}{x-y}$ into a series. When is the series finite?

2. Develop $\frac{x^m+y^m}{x+y}$ into a series.

3. Develop $\frac{a}{1-a}$. Ans. $a+a^2+a^3+\dots$

4. Develop $\frac{x}{1+x}$. Ans. $x-x^2+x^3-\dots$

5. Develop $\frac{a-bx}{a+cx}$.
Ans. $1-\frac{b+c}{a}x+\frac{c(b+c)}{a^2}x^2-\frac{c^2(b+c)}{a^3}x^3+\dots$

7. Develop $\frac{1}{1-2x+x^2}$. Ans. $1+2x+3x^2+4x^3+5x^4+\dots$

8. Develop $\frac{1+x}{1-x}$. 9. Develop $\frac{a+x}{1+x-x^2}$.

10. Develop $\frac{1}{(a-x)^2}$. Ans. $\frac{1}{a^2}+\frac{2}{a^3}x+\frac{3}{a^4}x^2+\frac{4}{a^5}x^3+\dots$

The correctness of the work can always be tested as follows: Multiply the part of the series found by the denomi-

nator; to the product add the remainder, if any; the result should be equal to the numerator. If there is no remainder, the series is finite (see Ex. 1, 2). The law of the series should be clearly stated for each of these examples, though it is of no importance to remember the different laws.

459. II.—Method of Multiplication or Involution.

When the multiplication of two or more polynomials is indicated we may frequently expand or develop the expression into a series by performing the indicated operation.

Thus: $(4x^3+16)(x+2)=4x^4+8x^3+16x^2+32$, a series in which the coefficients increase in geometrical progression while the exponents decrease in arithmetical progression. Again,

$$(x^3-3x^2y+3xy^2-y^3)(x^3+3x^2y+3xy^2+y^3)=x^6-3x^4y^2+3x^2y^4-y^6.$$

We do not usually, however, discuss any series formed in this way except those formed by that special case of multiplication called *Involution*. In series formed by involution the law is striking and important; every such series may be regarded as a particular case of the binomial formula, since, by appropriate transformations, we may involve a polynomial of any number of terms by this formula. Involution is an operation that can always be performed when the exponent of the power indicated is positive and entire; in this case, then, we expect and find that the series is finite. When, however, the exponent is negative, the operation is no longer generally possible; and in this case, therefore, we expect and find the series to be infinite. When the exponent indicating the operation is a fraction, the case falls under III.

EXAMPLES.

Expand the following expressions into series:

- | | |
|-------------------|-----------------------------|
| 1. $(x+y)^m$. | 2. $\frac{5}{(x+y+z)^2}$. |
| 3. $(x-y)^{-m}$. | 4. $(3x-9)^{\frac{1}{2}}$. |

$$5. \frac{a}{(b-x)^3} = a(b-x)^{-3}. \quad 6. \frac{ax}{(2x-5a)^5}.$$

460. III.—Method of Evolution.

The laws of coefficients and exponents in the binomial formula are always exactly true. Hence, no matter what expression is expanded by use of this formula, the result will be a series conforming to the laws just referred to; when the exponent indicating the operation is a fraction this series will be an expression for some root of the given polynomial. If this root can be extracted the series will be finite; if it cannot be extracted the series will be infinite. The direct application of the rules for Evolution will also give series; finite, when the polynomials operated on are perfect powers of the degree indicated, in other cases infinite; for the results obtained by these rules are the same as would be obtained by developing the given expressions by the binomial formula.

The methods of Involution and Evolution might both be included under the general name of the METHOD BY USE OF THE BINOMIAL FORMULA, if in operating on polynomials, these be placed in the *form* of binomials.

EXAMPLES.

Develop the following expressions into series:

$$1. (a^2 + x^2)^{\frac{1}{2}}. \quad \text{Ans. } a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} \dots$$

$$2. (c^2 - x^2)^{\frac{3}{2}}. \quad \text{Ans. } c^{\frac{3}{2}} \left(1 - \frac{3}{2c^2} x^2 - \frac{3}{2c^4} x^4 - \frac{5}{2^1 c^6} x^6 \dots \right)$$

$$3. (ab + x)^{\frac{2}{3}}.$$

$$4. \sqrt[3]{\frac{x^3}{x^3 + y^3}}. \quad \text{Ans. } 1 - \frac{y^3}{3x^3} + \frac{2y^6}{9x^6} - \frac{14y^9}{81x^9} + \dots$$

$$5. (1+x)^{\frac{1}{3}}. \quad \text{Ans. } 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \dots$$

6. $\sqrt{a^2+x}$. Applying the rule of Article 218,

$$\begin{array}{r}
 a^2+x \quad \left| a+\frac{x}{2a}-\frac{x^2}{8a^3}+\frac{x^3}{16a^5}-\frac{5x^4}{128a^7}+\dots \right. \\
 \hline
 a^2 \\
 2a+\frac{x}{2a} \quad \left| x+\frac{x^2}{4a^2} \right. \\
 \hline
 2a+\frac{x}{a}-\frac{x^2}{8a^3} \quad \left| -\frac{x^2}{4a^2} \right. \\
 \hline
 \frac{x^2}{4a^2}-\frac{x^2}{8a^3}+\frac{x^4}{64a^5} \\
 \hline
 2a+\frac{x}{a}-\frac{2x^2}{8a^3}+\frac{x^3}{16a^5} \quad \left| \frac{x^3}{8a^4}-\frac{x^4}{64a^5} \right. \\
 \hline
 \frac{x^3}{8a^4}+\frac{x^4}{16a^5}-\frac{x^5}{64a^5}+\frac{x^6}{256a^{10}} \\
 \hline
 \phantom{\frac{x^3}{8a^4}}-\frac{5x^4}{64a^5}+\frac{x^5}{64a^5}-\frac{x^6}{256a^{10}}
 \end{array}$$

Or, the series may be written,

$$a + \frac{x}{2a} - \frac{x^2}{2 \cdot 4a^3} + \frac{3x^3}{2 \cdot 4 \cdot 6a^5} - \frac{3 \cdot 5x^4}{2 \cdot 4 \cdot 6 \cdot 8a^7} + \dots$$

Or, it may be obtained directly by the Binomial Formula.

If $a=1$ and $x=1$, we have,

$$\sqrt{2} = \sqrt{1+1} = 1 + \frac{1}{2} - \frac{1}{8} + \frac{1}{16} - \dots, \text{ etc.}$$

$$7. \sqrt[3]{6} = (8-2)^{\frac{1}{3}}. \quad \text{Ans. } 2 - \frac{1}{2 \cdot 3} - \frac{1}{2^2 \cdot 3^2} - \frac{5}{2^3 \cdot 3^3} - \dots$$

$$8. \frac{x^2}{x^2+2ax+a^2}. \quad \text{Ans. } 1 - \frac{2a}{x} + \frac{3a^2}{x^2} - \frac{4a^3}{x^3} + \dots$$

$$9. (a^2-x)^{-\frac{1}{2}}. \quad \text{Ans. } \frac{1}{a} + \frac{x}{2a^3} + \frac{1 \cdot 3x^2}{2 \cdot 4a^5} + \frac{1 \cdot 3 \cdot 5x^3}{2 \cdot 4 \cdot 6a^7} + \frac{1 \cdot 3 \cdot 5 \cdot 7x^4}{2 \cdot 4 \cdot 6 \cdot 8a^9} + \dots$$

$$10. \sqrt{\frac{a^2+x^2}{a^2-x^2}}. \quad \text{Ans. } 1 + \frac{x^2}{a^2} + \frac{x^4}{2a^4} + \frac{x^6}{2a^4} + \dots$$

$$11. \frac{m}{\sqrt{a^2-x^2}}. \quad \text{Ans. } \frac{m}{\sqrt{a}} \left(1 + \frac{x^2}{4a^3} + \frac{1 \cdot 5x^4}{4 \cdot 8a^5} + \frac{1 \cdot 5 \cdot 9x^6}{4 \cdot 8 \cdot 12a^7} + \dots \right)$$

461. IV.—Method of Indeterminates' Coefficients.
This method is founded on certain properties of identical equations.

An Identical Equation (344) is one that is satisfied by the substitution of any quantities whatever for the unknown quantities entering it. It differs both from the common equation, which may be satisfied only by a limited number of substitutions for the unknown quantity, and from the indeterminate equation, satisfied by an infinite number of sets of values of the unknown quantities entering it. Each of the unknown quantities in an identical equation may have an infinite number of values; i. e., each of these unknown quantities is indeterminate. The coefficients in any identical equation, being coefficients of indeterminates, are generally called Indeterminate or Undetermined Coefficients; the use of the term Indeterminates' Coefficients, has been suggested by Prof. W. T. Welcker, and this name will be here adopted, as it is free from the principal objections urged against the other two forms of expression.

462. Proposition I.—*In an identical equation, involving but one unknown quantity, the coefficients of the like powers of this unknown quantity in the two members are separately equal to each other.*

Let $A+Bx+Cx^2+Dx^3+\dots=A'+B'x+C'x^2+D'x^3+\dots(1)$ represent any identical equation with but one unknown quantity, the coefficients being entirely independent of this quantity. Since the equation is true for any value that may be assigned to x , it must be true when $x=0$. Making this substitution in (1), we find $A=A'$. As the coefficients are independent of x , they have the same value, no matter what the values of x . A , then, always equals A' .

Hence, from (1),

$$Bx+Cx^2+Dx^3+\dots=B'x+C'x^2+D'x^3+\dots(2);$$

or, dividing both members by x ,

$$B+Cx+\dots=B'+C'x+\dots(3);$$

whence, by the same reasoning as before, $B=B'$. Sub-

tracting, then, the first term from each member of (3), we obtain, $Cx + Dx^2 + \dots = C'x + D'x^2 + \dots$ (4); or, dividing both members by x , $C + Dx + \dots = C' + D'x + \dots$ (5); whence, reasoning as before, $C = C'$. Continuing this process, we may show that $D = D'$, $E = E'$, etc. The truth of the proposition is evidently not affected if one or more of the coefficients reduce to 0.

463. Corollary.—*In any identical equation of which one member is 0, the equation involving but one indeterminate, the coefficients of the different powers of this quantity are separately equal to zero.*

Let $A + Bx + Cx^2 + Dx^3 + \dots = 0$ (1), be such an equation, A , B , C , etc., being entirely independent of x . Then is $A = 0$, $B = 0$, $C = 0$, and so on. (1) may be placed under the form: $A + Bx + Cx^2 + Dx^3 + \dots = 0 + 0.x + 0.x^2 + 0.x^3 + \dots$; whence, by the main theorem, $A = 0$, $B = 0$, $C = 0$, etc.

464. Proposition II.—*In any identical equation, containing any number of indeterminates, the coefficients of like powers of each indeterminate, in the two members, are equal.*

Let

$$\left. \begin{array}{l} A + Bx + Cx^2 + Dx^3 + \dots \\ + My + Ny^2 + Py^3 + \dots \\ + Fz + Gz^2 + Hz^3 + \dots \\ + \text{etc.} \end{array} \right\} = \left\{ \begin{array}{l} A' + B'x + C'x^2 + D'x^3 + \dots \\ + M'y + N'y^2 + P'y^3 + \dots \\ + F'z + G'z^2 + H'z^3 + \dots \\ + \text{etc.} \end{array} \right.$$

represent any identical equation, the coefficients being entirely independent of x , y , z , etc.

Since the equation is true for all values of x , y , z , etc., it is true when y and z , etc., $= 0$. Making the substitution thus suggested in (1), we obtain

$$A + Bx + Cx^2 + Dx^3 + \dots = A' + B'x + C'x^2 + \dots \quad (2),$$

whence, (462) $A = A'$, $B = B'$, $C = C'$, and so on. Again, the equation, being identical, is true when all the indeterminate quantities, except y , $= 0$. Hence,

$$My + Ny^2 + Py^3 + \dots = M'y + N'y^2 + P'y^3 + \dots;$$

whence, (462) $M = M'$, $N = N'$, etc. In like manner we may prove that $F = F'$, $G = G'$, etc.

465. Corollary.—*In any identical equation, involving any number of indeterminates, and in which either member is 0, the coefficients of the different powers of each indeterminate are separately equal to 0.*

The principle stated in its most general form in Proposition II. is called the **Principle of Indeterminates' Coefficients**, and is exceedingly valuable in the important operation of developing expressions into series.

466. To develop an expression into a series by the method of Indeterminates' Coefficients.

Let it be required to develop $\frac{a}{a'+b'x}$ into a series, arranged according to the ascending powers of x . If the indicated division were performed, the first term of the quotient would involve the zero power of x . Place, then, the fraction equal to a series beginning with x^0 , thus;

$\frac{a}{a'+b'x} = P + Qx + Rx^2 + Sx^3 + \dots \dots \dots (1)$, the coefficients P, Q, R , etc., being independent of x , and to be determined so that the development will be true for all values of x . The development being merely the result of the division indicated in the first member, (1) must be an identical equation, P, Q, R , etc., depending only on a, a' and b' for their values. Clearing (1) of fractions, we obtain,

$$\begin{array}{r|l} a = a'P + a'Q & x + a'R \mid x^2 + a'S \mid x^3 + \dots \dots \\ + b'P & + b'Q \mid + b'R \end{array}$$

whence, (462) $a = a'P \therefore P = \frac{a}{a'}$; $0 = a'Q + b'P = a'Q + b' \frac{a}{a'} \therefore$

$$a'Q = -\frac{ab'}{a'} \therefore Q = -\frac{ab'}{a'^2}. \text{ Similarly, } R = \frac{ab'^2}{a'^3}, \text{ etc.}$$

$$\text{Hence the series is: } \frac{a}{a'} - \frac{ab'}{a'^2}x + \frac{ab'^2}{a'^3}x^2 - \frac{ab'^3}{a'^4}x^3 + \dots$$

If the expression is of the form $\frac{a}{a'+b'x}$, or, in general, such that, on performing the indicated operation, the first term of the result will involve only the zero power of the indeterminate, we may develop the expression into

a series arranged according to the powers of this quantity by the

RULE.—Place the given expression equal to a development of the form $P+Qx+Rx^2+Sx^3+\dots$. Clear the equation of fractions, and equate the coefficients of like powers of x in the two members. From the equations thus obtained find the values of P, Q, R , etc., and substitute these values for P, Q, R , etc., in the assumed development.

467. If the given expression is not of such a form that, on carrying out the indicated operation, x^0 will be the only power of x in the first term, we must assume a different series from that indicated in the rule. The most general form of a fraction to be developed into a series, is

$$\frac{ax^m+bx^{m+1}+cx^{m+2}+\dots}{a'x^n+b'x^{n+1}+cx^{n+2}+\dots}$$

If the operation here indicated were performed, the first term of the quotient would be x^{m-n} . If $m=n$, $x^{m-n}=x^0$, and we develop as directed in the rule; if $m>n$, $m-n$ is positive, and if, in assuming a development, we begin at x^0 , some of the coefficients will be 0, and as this does not affect the truth of the main principle, the application of the rule will give us the true development here also, P, Q , etc., reducing to 0, until we arrive at the coefficient of x^{m-n} . If, however, $m<n$, $m-n$ is negative, and in assuming a development $P+Qx+Rx^2+\dots$, we assume all the coefficients of powers of x below the zero power to be separately 0, which is untrue, and as we begin with an untruth, the mathematical work will point out this fact by leading us into some palpable absurdity, if we attempt to develop in this case by the rule (466). As the only error consists in assuming the powers of x to begin where they do not, if we assume them to begin at the proper point (in this case at the $(m-n)$ th power), there will be no further error, and, clearing of fractions, etc., as in the rule, we have the series required. The development assumed would then be $Px^{m-n}+Qx^{m-n+1}+\dots+S+Tx+Vx^2+\dots$

468. An expeditious method of determining the form of series to be adopted is the following: Place the unknown quantity in the given expression equal to 0. If, under this supposition, the given expression reduces to a finite quantity, the first term of the development will not contain x , and the development will be of the form $P+Qx+Rx^2+\dots$. If the expression reduces to 0, the first term of the series will contain x with a positive exponent, and if we assume the development $P+Qx+Rx^2+\dots$, some of the first coefficients will reduce to 0. If, when $x=0$, the given expression reduces to the form $\frac{M}{0}=\infty$, the first term of the development must contain x with a negative exponent.

The reason of this method of procedure is apparent on examining the general form, $\frac{ax^n+bx^{n+1}+cx^{n+2}+\dots}{a'x^m+b'x^{m+1}+c'x^{m+2}+\dots}$.

Dividing both terms by x^n , we obtain $\frac{a+bx+cx^2+\dots}{a'x^{m-n}+b'x^{m-n+1}+c'x^{m-n+2}+\dots}$. If, in this, the value $x=0$ be substituted, the expression reduces to $\frac{a}{a'}$, when $m=n$; to $\frac{a}{0}$, when $m>n$; and to $\frac{0}{a'}$, when $m<n$, since, in this case, $m-n$ is negative, and $x^{m-n}=\frac{1}{x^{n-m}}$,

whence, $a'x^{m-n}=\frac{a'}{x^{n-m}}=\frac{a'}{0}=\infty$ when $x=0$, and

$$a \div \frac{a'}{0} = \frac{a \cdot 0}{a'} = \frac{0}{a'} = 0.$$

469. If the given expression is of the form $\sqrt[n]{a+bx}$, we may place it equal to a development of the assumed form, $P+Qx+Rx^2+Sx^3+\dots$, raise both members to the n th power, and equate the coefficients of like powers of the indeterminate in the two members in the resulting identical equation, and from the equations thus obtained, deduce the values of P , Q , R , etc.

Take the following example: Develop $\sqrt{(a^2+x^2)^3}$ into a series. Assuming the development, we find,

$$(a^2+x^2)^{\frac{3}{2}}=P+Qx+Rx^2+Sx^3+Tx^4+\dots(1).$$

Squaring both members of (1),

$$(a^2+x^2)^2 = P^2 + \begin{array}{c} PQ \\ +PQ \end{array} x + \begin{array}{c} PR \\ +Q^2 \\ +PR \end{array} x^2 + \begin{array}{c} PS \\ +QR \\ +PS \end{array} x^3 + \dots (2).$$

Cubing a^2+x^2 , and reducing,

$$a^3+3a^2x+3a^2x^2+x^3=P^3+2PQx+(2PR+Q^2)x^2+\dots (3).$$

Equating coefficients, (462), $P^2=a^2 \therefore P=a$; $2PQ=0$; or

$$2a^2Q=0 \therefore Q=0; 2PR+Q^2=3a^2 \therefore 2PR=3a^2; 2a^2R=3a^2,$$

$$\therefore R=\frac{3}{2}a. \text{ Similarly, } T=\frac{3}{8}a, V=-\frac{1}{16}a, \text{ etc. Hence, the}$$

required series is $a^3+\frac{3}{2}ax^2+\frac{3}{8}a^2x^4-\frac{1}{16}a^3x^6+\dots$. This could, of course, have been obtained at once by the application of the Binomial Formula.

Again, had we placed $(a^2+x^2)^{\frac{3}{2}} = -P-Qx-Rx^2-\dots$, and squared, we should have obtained the same result, numerically. Hence, as the coefficients may be either positive or negative, the series may be written,

$$\pm \left(a^3 + \frac{3}{2}ax^2 + \frac{3}{8}a^2x^4 - \frac{1}{16}a^3x^6 + \dots \right).$$

If, on the other hand, we have an expression of the form $(a+bx+\dots)^n$, we may, evidently, place it equal to $P+Qx+Rx^2+\dots$, and, expanding the first expression, and equating the coefficients, find the values of P, Q, R , etc. If we have $(a+bx+\dots)^{\frac{k}{n}}$, we may place it equal to a development of this form, raise both members to the n th power, and then expand and proceed as before. This is illustrated in the preceding example. If n is even and k odd, the development will have the double sign; if both n and k are even, the exponent $\frac{k}{n}$ is not in its simplest form. If n is odd, there is no double sign, as a root of an odd degree can have but one sign.

When, however, an expression may be developed by the binomial formula, it is best to apply this formula at once. The method by indeterminates' coefficients, while not

always the most expeditious, is the most general and important method of developing expressions into series. In fact, the binomial formula itself, in its general expression, is usually deduced by the use of this method, the only one of the four enumerated in (457) that is *always* applicable.

EXAMPLES.

Develop the following expressions into series:

$$1. \frac{1-x}{2x^2+3x^3}. \quad \text{Ans. } \frac{1}{2x^2} - \frac{5}{4x} + \frac{15}{8} - \frac{45}{16}x + \frac{135}{32}x^2 \dots$$

See Article (467).

$$2. \frac{1}{1-2x+x^2}. \quad \text{Ans. } 1+2x+3x^2+4x^3+5x^4+\dots$$

$$3. \frac{1-x}{1+x+x^2}. \quad \text{Ans. } 1-2x+x^2+x^3-2x^4+x^5+x^6-\dots$$

$$4. \frac{x^4-3x^2+2}{x^2-x+1}. \quad \text{Ans. } 2+2x-3x^2-5x^3-x^4 \dots$$

$$5. \frac{1-2x+3x^2}{x^2+2x^3+3x^4}. \quad \text{Ans. } \frac{1}{x^2} - \frac{4}{x} + 8 - 4x - 16x^2 + 44x^3 \dots$$

$$6. \frac{3-x^2}{x-2x^2}. \quad \text{Ans. } \frac{3}{x} + 5x + 16x^2 + 20x^3 + 40x^4 + \dots$$

$$7. \frac{1-x+x^2-x^3+x^4-\dots}{1+x+x^2+x^3+x^4+\dots}. \quad \text{Placing this equal to the de-}$$

velopment of the usual form, we obtain,

$$\frac{1-x+x^2-x^3+x^4-\dots}{1+x+x^2+x^3+x^4+\dots} = P + Qx + Rx^2 + Sx^3 + \dots (1);$$

clearing of fractions,

$$\begin{aligned} 1-x+x^2-x^3+x^4-\dots &= P + Qx + Rx^2 + Sx^3 + \dots \\ &+ Px + Qx^2 + Rx^3 + \dots \\ &+ Px^2 + Qx^3 + \dots \\ &+ Px^3 + \dots; \end{aligned}$$

whence, $P=1$; $Q+P=-1$, $\therefore Q=-2$; $P+Q+R=1 \therefore R=2$, etc; hence,

$$\frac{1-x+x^2-x^3+x^4-\dots}{1+x+x^2+x^3+x^4+\dots} = 1-2x+2x^2-2x^3+2x^4 \dots$$

$$8. \sqrt{\frac{1+x^2}{1-x^2}}. \quad \text{Ans. } \pm \left(1+x^2 + \frac{x^4}{2} + \frac{x^6}{2} + \frac{3x^8}{8} + \dots \right)$$

$$9. \frac{6+3x+2x^2}{9-x+2x^2}. \quad \text{Ans. } \frac{2}{3} + \frac{11}{27}x + \frac{29}{243}x^2 - \frac{169}{2187}x^3 - \dots$$

$$10. \sqrt{\frac{1+2x}{1-x+x^2}}; \quad \frac{1+2x+3x^2+4x^3+5x^4}{2+4x+6x^2+8x^3+10x^4+12x^5}.$$

470. It will be seen that the methods of developing expressions into series depend on the fact that the series is the result of the performance of certain operations, indicated in the given expression. When the operations indicated can be completely and exactly performed, the resulting series will be *finite*; when, on the other hand, the operations indicated cannot be completely and exactly performed, the series will be *infinite*. The development of expressions into series is an important and useful operation; it will often be found much more simple to operate upon a series derived from a given expression than upon the expression itself.

471. **Generation of Recurring Series.**—One of the most important classes of series exemplified in the foregoing discussions is the **Recurring Series**. If the development of $\frac{a}{a'+b'x} = \frac{a}{a'} - \frac{ab'}{a'^2}x + \frac{ab'^2}{a'^3}x^2 - \frac{ab'^3}{a'^4}x^3 + \frac{ab'^4}{a'^5}x^4 - \dots$ be examined, the fact that any term after the first may be obtained by multiplying the preceding one by $-\frac{b'}{a'}x$, becomes apparent. The series, then, is *recurring*, and as the scale, $-\frac{b'}{a'}x$, contains but one term, the series is of the first order; it is a geometrical progression. The higher orders of recurring series may be considered as bearing the same relation to the geometrical progression that the higher arithmetical progressions bear to the ordinary arithmetical progression. If, now, $\frac{a+bx}{a'+b'x+c'x^2}$ be developed into a

series, we shall obtain

$$\frac{a+bx}{a'+b'x+c'x^2} = P+Qx+Rx^2+\dots(1);$$

or, clearing of fractions, and equating coefficients,

$$Pa'=a \therefore P=\frac{a}{a'}; Qa'+Pb'=b \therefore Q=-\frac{b'}{a'}P+\frac{b}{a'};$$

$$Ra'+Qb'+Pc'=0 \therefore R=-\frac{b'}{a'}Q-\frac{c'}{a'}P;$$

$$Sa'+Rb'+Qc'=0 \therefore S=-\frac{b'}{a'}R-\frac{c'}{a'}Q.$$

In this development, then, each coefficient, commencing at the third, is obtained by taking the algebraic sum of the products of the immediately preceding one by $-\frac{b'}{a'}$, and of the second preceding by $-\frac{c'}{a'}$. In this recurring series, therefore, $-\frac{b'}{a'}, -\frac{c'}{a'}$, is the scale of the coefficients.

From this scale of coefficients, and the sequence of the powers of x in (1), we see that any term of the series will be obtained by multiplying the term immediately preceding by $-\frac{b'}{a'}x$, and the second preceding by $-\frac{c'}{a'}x^2$, and taking the algebraic sum of these products. The series is, then, a recurring series of the second order.

An examination of the development,

$$\frac{a+bx+cx^2}{a'+b'x+c'x^2+d'x^3} = P+Qx+Rx^2+Sx^3+\dots(1),$$

or, clearing of fractions,

$$a+bx+cx^2=a'P+\frac{b'P}{+a'Q}x+\frac{c'P}{+a'R}x^2+\frac{d'P}{+a'S}x^3+\frac{d'Q}{+a'T}x^4+\dots$$

etc., etc.,

shows that, as

$$Pa'=a \therefore P=\frac{a}{a'}; b'P+a'Q=b \therefore Q=-\frac{b'}{a'}P+\frac{b}{a'};$$

$$c'P+b'Q+a'R=c \therefore R=-\frac{b'}{a'}Q-\frac{c'}{a'}P+\frac{c}{a'};$$

the first; when we have passed over the third term, we have passed over *two* after the first, and have added δ twice; at the fifth term we have passed over *four* after the first, and have added δ *four times*; at the *tenth* term we have added δ *nine times*; finally, after passing over the *n*th term we have passed over $n-1$ terms after the first, and hence, δ must have been added $n-1$ times; i. e., the *n*th term is $a + (n-1)\delta$. If this term be denoted by l , we have $l = a + (n-1)\delta$, a formula expressing the relations of a , l , n , and δ , for any arithmetical progression of the first order; by means of this formula we may find any one of the quantities, a , l , n , δ , if the other three are given. See Table I.

If $n = \infty$, $l = \pm \infty$, as δ is positive or negative.

The Higher Arithmetical Progressions will be discussed hereafter.

475.—II. Harmonical Progressions.

Let $a', b', c', d' \dots (1)$ represent any harmonical progression. Taking the reciprocals of the terms, we obtain, $\frac{1}{a'} \dots \frac{1}{b'} \dots \frac{1}{c'} \dots \frac{1}{d'} \dots (2)$, an arithmetical progression. If, now, l' denote the *n*th term, and δ' the common difference, of (2), $l' = \frac{1}{a'} + (n-1)\delta'$ (474). Hence, if λ represent the *n*th term of (1), $\lambda = \frac{1}{l'} = \frac{a'}{a'(n-1)\delta' + 1}$, or, replacing δ' by its value, $\frac{1}{b'} - \frac{1}{a'} = \frac{a' - b'}{a'b'}$, $\lambda = \frac{a'b'}{b' + (n-1)(a' - b')}$.

476.—III. Geometrical Progressions.

In the common geometrical progression: Any term equals the first, multiplied by that power of the ratio, of which the exponent is the number of terms preceding the required term.

Let $a:b:c:d:\dots$ be a geometrical progression of which the ratio is ρ . Then, (450) $b = a\rho$, $c = b\rho = a\rho^2$, $d = c\rho = a\rho^3$, $\dots$; the exponent of ρ in the value of each term being the number of terms preceding the one in

question. The $(n-1)$ th term, as k , will be $a\rho^{n-2}$, if the law that is evident in the formation of a, b, c, d , holds good generally. Now, suppose $k=a\rho^{n-2}$; then, (450), the next term must be $a\rho^{n-2}\rho=a\rho^{n-1}$; or, if l denote the n th term, $l=a\rho^{n-1}$. Hence, if the law just referred to holds good in the formation of any term, it will hold good, also, in the formation of the next term. As it holds good for the formation of the fourth term, it must hold good for the formation of the fifth, and, consequently, for the formation of the sixth, and so on, i. e., it is general.

The proposition (476) may be expressed in the formula, $l=a\rho^{n-1}$, expressing the relations of a, l, ρ , and n for any geometrical progression, and enabling us to find any one, if the other three are given. See Table II.

If $n=\infty$, $l=\pm\infty$, or ± 0 , as a is positive or negative, and $\rho > 1$, or < 1 .

477. IV.—Binomial Formula.

Any term may be obtained by applying the laws of coefficients and exponents (201-2), or, directly, by the proper substitutions in the GENERAL TERM.

The form of the last term of this series is always the same as the form of the first term.

478.—V. Recurring Series.

If n is not very large, the n th term may be found by a sufficient extension of the development. Any term may also be found by direct application of the law, if this be sufficiently explicit and detailed. Finally, the scale may be found, and if the number of terms given is at least twice that in the scale, any term may be found by application of the scale.

To Find the Scale of the Series.

The method given in (471), of finding the scale of a recurring series, is useful only when the generating fraction is given. If the series only is given, and the scale is required, the following method may be employed:

form-
the suc-

1°. If the series is of the first order, the scale may be found by dividing any term by its immediate predecessor, since each term is found by multiplying its immediate predecessor by the scale.

2°. If the series is of the second order, $a, b, c, d, e, \dots$ representing the terms, let s, s' represent the scale. Then, from the definition, $c=bs+as'$, and $d=cs+bs'$, whence the values of s and s' in terms of a, b, c, d , etc., may be found by the ordinary method of solving simultaneous equations of the first degree. $s' = \frac{c^2-bd}{ac-b^2}$, $s = \frac{ad-bc}{ac-b^2}$.

3°. If the series is of the third order, $a, b, c, d, e, f, \dots$ being the given terms, and s, s', s'' , the scale; $d=sc+s'b+s''a$, $e=sd+s'c+s''b$, $f=se+s'd+s''c$, from which the values of s, s', s'' may be found.

The same method may evidently be extended to series of any degree, requiring the solution of as many simultaneous equations of the first degree as there are terms in the scale. The results may always be tested by applying them to the given series. To employ this method, begin by assuming the series to be of the first order, and if the scale thus obtained does not apply to the series, assume the series to be of the second order, and so proceed until a scale is found that may be applied to the given series. The opposite mode of procedure may also be adopted, viz.: assuming the series to be of a high order, and working downward. If we assume the order too high, some terms of the scale will reduce to 0; if too low, the scale found will not apply to the given series.

The application of this process forms a test by which the recurrence of a series may be ascertained. It cannot be definitely concluded, however, that, since a finite portion of an infinite series presents no recurrence, therefore the series itself is not recurring.

Before taking up the second case, the following proposition is to be demonstrated.

479. Lemma.—*The first term of the n th order of differences of any series is,*

$$\pm \left(a - nb + \frac{n(n-1)}{1.2}c - \frac{n(n-1)(n-2)}{1.2.3}d + \dots \right),$$

the upper sign being taken when n is even, the lower sign when n is odd.

Let $a, b, c, d, e, f, g, h, \dots$, represent any series whatever. The successive orders of differences are:

$$b-a, c-b, d-c, e-d, f-e, g-f, \dots \quad (1)$$

$$c-2b+a, d-2c+b, e-2d+c, f-2e+d, g-2f+e, \dots \quad (2)$$

$$d-3c+3b-a, e-3d+3c-b, f-3e+3d-c, g-3f+3e-d, \dots \quad (3)$$

$$e-4d+6c-4b+a, f-4e+6d-4c+b, g-4f+6e-4d+c, \dots \quad (4)$$

$$f-5e+10d-10c+5b-a, g-5f+10e-10d+5c-b, \dots \quad (5)$$

$$g-6f+15e-20d+15c-6b+a, \dots \dots \dots \quad (6)$$

Denoting, by D_1, D_2, D_3, D_4 , etc., the first terms of the first, second, third, etc., orders of differences, and changing the sequence in each case, so as to bring a first, we have:

$$D_1 = -a+b, \quad D_2 = a-2b+c,$$

$$D_3 = -a+3b-3c+d, \quad D_4 = a-4b+6c-4d+e,$$

$$D_5 = -a+5b-10c+10d-5e+f, \quad \dots \dots \dots$$

$$D_6 = a-6b+15c-20d+15e-6f+g.$$

In these equations the coefficients in the second members are the same as in the binomial formula, the coefficients in the value of D_1 being the same as in the first power of a binomial; of D_2 , the same as in the second power, and so on, the sub of D taking the place of the exponent of the power. (This of the numerical values of the coefficients.) Again, if, when the sub is odd, a minus sign be prefixed to the values, *e. g.*, for D_1, D_3, D_5 , etc., all the coefficients will conform to the law of signs of the coefficients of that power of a residual whose exponent is written as a sub to D . The unit parts of the terms consist of the terms of the given series, in their proper sequence, *i. e.*, a in the first term, b in the second, and so on.

Now let us assume these laws to hold good for the formation of the first n orders of differences. Then, as the suc-

cessive terms in each order are of the same general form as the first, the n th order of differences will be

$$\begin{aligned} &\pm \left(a - nb + \frac{n(n-1)}{1.2}c - \frac{n(n-1)(n-2)}{1.2.3}d + \dots \right), \\ &\pm \left(b - nc + \frac{n(n-1)}{1.2}d - \frac{n(n-1)(n-2)}{1.2.3}e + \dots \right), \dots \end{aligned}$$

Hence, the first term of the $(n+1)$ th order of differences, obtained by subtracting the first of these expressions from the second, will be,

$$\mp \left(a - (n+1)b + \left(\frac{n(n-1)}{1.2} + n \right) c - \left(\frac{n(n-1)(n-2)}{1.2.3} + \frac{n(n-1)}{1.2} \right) d + \dots \right)$$

the upper or lower sign being taken according as n is even or odd.

$$\begin{aligned} \text{Now, } \frac{n(n-1)}{2} + n &= \frac{n^2 - n + 2n}{2} = \frac{n^2 + n}{2} = \frac{(n+1)n}{2}; \\ \frac{n(n-1)(n-2)}{1.2.3} + \frac{n(n-1)}{1.2} &= \frac{n^3 - 3n^2 + 2n + 3n^2 - 3n}{1.2.3} \\ &= \frac{n^3 - n}{1.2.3} = \frac{(n+1)n(n-1)}{1.2.3}. \end{aligned}$$

The first term, then, of the $(n+1)$ th order of differences is $\mp \left[a - (n+1)b + \frac{(n+1)n}{1.2}c - \frac{(n+1)n(n-1)}{1.2.3}d + \dots \right] = D_{n+1}$, which evidently conforms to the same law of coefficients and unit parts of terms, as the first term of the n th order. Hence, *If the first term of the n th order of differences is of the form $\pm \left(a - nb + \frac{n(n-1)}{1.2}c - \dots \right)$ the first term of the $(n+1)$ th order is likewise of this form.* That is, if the laws already given apply to the formation of the first term of any order of differences, they apply to the formation of the first term of the next higher order. We have shown them applicable to the formation of the first term of the sixth order; they are therefore applicable to the formation of the first term of the seventh order. Hence, the first term of the eighth order conforms to these laws, and so on. Hence, the first term of the n th order will be

$$\pm \left[a - nb + \frac{n(n-1)}{1.2}c - \frac{n(n-1)(n-2)}{1.2.3}d + \dots \right] = D_n.$$

480. Scholium.—To find the first term of the n th order of differences, $n+1$ terms of the original series are necessary.

(B.) THE GENERAL LAW OF THE SERIES IS UNKNOWN.

481. *To find the n th term of any series.*

Let $a, b, c, d, e, f, g, \dots$ represent any given series. Forming the successive orders of differences, we obtain,

- 1°. $b-a, c-b, d-c, e-d, f-e, g-f, \dots$
 - 2°. $c-2b+a, d-2c+b, e-2d+c, f-2e+d, \dots$
 - 3°. $d-3c+3b-a, e-3d+3c-b, f-3e+3d-c, g-3f+3e-d, \dots$
 - 4°. $e-4d+6c-4b+a, f-4e+6d-4c+b, g-4f+6e-4d+c, \dots$
 - 5°. $f-5e+10d-10c+5b-a, g-5f+10e-10d+5c-b, \dots$
 - 6°. $g-6f+15e-20d+15c-6b+a, \dots$
- etc., etc., etc.

Letting D_1 denote the first term of the first order of differences, D_2 the first term of the second, D_3 the first term of the third order, etc.,

$$\begin{aligned} D_1 &= b-a \therefore b = a + D_1 \\ D_2 &= c-2b+a \therefore c = -a + 2b + D_2 = a + 2D_1 + D_2 \\ D_3 &= d-3c+3b-a \therefore d = a - 3b + 3c + D_3 = a + 3D_1 + 3D_2 + D_3 \\ D_4 &= e-4d+6c-4b+a \therefore e = -a + 4b - 6c + 4d + D_4 \\ &= a + 4D_1 + 6D_2 + 4D_3 + D_4 \\ D_5 &= f-5e+10d-10c+5b-a \\ &\therefore f = a + 5D_1 + 10D_2 + 10D_3 + 5D_4 + D_5 \\ D_6 &= g-6f+15e-20d+15c-6b+a \\ &\therefore g = a + 6D_1 + 15D_2 + 20D_3 + 15D_4 + 6D_5 + D_6 \\ &\dots\dots\dots \end{aligned}$$

$$D_n = \pm \left(a - nb + \frac{n(n-1)}{2}c - \frac{n(n-1)(n-2)}{1.2.3}d + \dots \right)$$

In each of these equations expressing the values of $b, c, d, e, \dots$ the coefficients are the same as in the binomial formula: in the value of the second term, b , the coefficients are the same (1, 1) as in the first power of a binomial; in the value of the third term, c , they are (1, 2, 1) the same as in the second power of a binomial; in the value of the fourth term, the same as in the third power of a binomial, and so on. Again, the first term, in each case, is the first

term of the assumed series; the subs of D are the same as the exponents of the powers of the *second term* of a binomial: in the value of b , the second term, the sub, 1, is the same as the exponent of y in the *first* power of $(x+y)$; in the value of the third term, the subs, 1, 2, are the same as the exponents of y in the *second* power of $(x+y)$; in the value of the fourth term, the subs are the same as the exponents of y in the development of the *third* power of $(x+y)$, and so on. As the successive orders of differences are all obtained in the same way, and the values of the terms of the series are obtained from the first terms of the successive orders of differences, it is clear that the same law that holds good for the formation of the values of the first, second, third, . . . seventh terms will hold good for the value of any term of the series. Hence the $(n+1)$ th term must be,

$$T = a + nD_1 + n \frac{n-1}{1.2} D_2 + \dots + \frac{n(n-1)(n-2) \dots 1}{1.2.3 \dots n} D_n; \text{ or}$$

$$T = a + nD_1 + n \frac{(n-1)}{1.2} D_2 + \frac{n(n-1)(n-2)}{1.2.3} D_3 + \dots + D_n \quad (1).$$

482. If the terms of any order of differences, as the r th, be equal, the terms of the succeeding orders of differences will be 0, and the formula, extended to D_r , will give an exact result. If, before reaching D_n , the terms of no order are all equal, an exact result can be obtained only by taking all the terms of T , though an approximate result, nearer the truth the greater the number of terms taken, may be obtained by carrying out the formula to a sufficient number of terms to ensure the required degree of accuracy. In this case the formula is useless, as far as *exact* results are concerned, for in order to find D_n , $(n+1)$ terms of the series are necessary, and the fulfillment of this condition precludes the necessity of finding the $(n+1)$ th term—destroys the problem.

If only $s+1$ terms of the series be given, the formula,

$$T = a + nD_1 + \frac{n(n-1)}{1.2} D_2 + \dots + \frac{n(n-1)(n-2)(n-3) \dots (n-s+1)}{1.2.3.4 \dots s} D_s \quad (2).$$

must be used; i. e., (1) cannot be carried beyond D . The result given by (2) will approximate more closely to the truth, the nearer s is to n , and the more rapidly $D_{s+1}, D_{s+2}, \dots$ diminish in value.

The formula (1), then, is useful for exact results only in the case of arithmetical progressions, and the number of terms in the second member of the formula, is, for any particular case, greater by 1 than the number expressing the order of the progression. Thus, for the progression of the r th order, before referred to, $r+1$ terms must be used; for a progression of the first order, 2 terms are used; for one of the second order, 3 terms, etc. For a progression of the first order, we have, $T=a+nD_1$, or, since D_1 is, in this case, the common difference of the progression, $T=a+n\delta$, for the $(n+1)$ th term; and for the n th term, denoting this by l , $l=a+(n-1)\delta$, the same result previously obtained (474).

The formula (1) is of necessity useless, if the last term of an infinite series is required, except when the series is an arithmetical progression, in which, from the nature of the case, the last term will be either $\pm\infty$, as the series is increasing or decreasing.

EXAMPLES.

1. Find the n th term of the series, 1, 3, 5, 7....
2. Find the n th term of the series, 4, 7, 10, 13....; of the series 9, 16, 23, 30....
3. Find the n th term of the series 2, 4, 6, 8....; of the series 3, 9, 27, 81, 243....; of the series 5, 25, 125, 625, 3125....
4. Find the 13th term of the series 3, 12, 48, 192....; the 10th term of the series 5, 15, 45, 135....; the 11th term of the series 6, 18, 54, 162....

Ans. 50331648; 98415; 354294.

5. Find the 10th term of the series 1, 4, 8, 13, 19....; the 50th term; the 30th term of the series 1, 4, 9, 16....; the n th term.

Ans. 64; 1324; 900; n^2 .

6. Find the n th term of the series 1, 3, 6, 10....; of the series 1.2, 2.4, 4.5, 5.8, 7.10....

$$\text{Ans. } \frac{n(n+1)}{2}; \frac{n(n+1)(n+2)}{3}.$$

7. Find the 14th term of the series $x, 3x^2, 6x^3, 10x^4, \dots$; the 12th term of $x^2, 4x^3, 6x^4, 11x^5, 28x^6, 63x^7, \dots$; the 8th term of $1, 4x^3, 8x^5, 13x^7, \text{etc.}$ *Ans.* $105x^{14}; 6396x^{12}; 43x^8$.

8. Find the 8th term of $1, 2x, 8x^2, 28x^3, 100x^4, \dots$; the 12th term of $3, 5, 7, 13, 23, 45, \dots$ *Ans.* $4516x^8; 2733$.

9. Find the n th term of the series $2x^3, 6x^5, 12x^7, 20x^9, \dots$; of $1, 12x^4, 54x^5, 160x^{13}, \dots$; of $6x^3, 24x^5, 60x^9, 120x^{13}, \dots$

$$\text{Ans. } n(n+1)x^{2n+1}; \frac{n^3(n+1)}{2}x^{4(n-1)}; n(n+1)(n+2)x^{3n}.$$

NOTE.—The second series may be put in the form $1, 3.4x^4, 6.9x^5, 10.16x^{12}, \dots$; the third may be put in the form $1.2.3.x^3, 2.3.4x^5, 3.4.5x^9, \dots$

10. Find the n th term of $15x, 105x^2, 315x^3, 693x^7, \dots$; of $20x, 96x^3, 312x^5, 720x^{10}, \dots$

INTERPOLATION.

483. When terms are inserted between the terms, two and two, of a given series, the operation is called **Interpolation**, i.e., *placing between*.

A. THE LAW OF THE SERIES IS KNOWN.

484.—I. Arithmetical Progressions.

To insert, between the terms two and two, of an arithmetical progression, the same number of arithmetical means.

Let $a \cdots b \cdots c \cdots d \cdots e \cdots f \cdots$ be the given progression, and let m be the number of means to be inserted between a and b , b and c , c and d , etc. From the definition of arithmetical means, those inserted between a and b must, with a and b , make up a progression; similarly, those between b and c , taken with b and c , and so on. In each of these minor progressions, then, there will be $m+2$

terms. Taking the first, a will be the first term and b the $(m+2)$ th term, and denote by δ' the common difference. Then $b = a + (m+1)\delta'$, whence $\delta' = \frac{b-a}{m+1}$ ($m+1$ being the number of terms preceding b). Similarly, in the second minor progression $\div b \dots c$, δ'' being the common difference, $\delta'' = \frac{c-b}{m+1}$; in the next, $\delta''' = \frac{d-c}{m+1}$, and so on. The numerators of all of these fractions are equal, since each is equal to the common difference of the given progression, and, the denominators being equal, $\delta' = \delta'' = \delta'''$, etc. Hence, as the last term of each minor progression is the first term of the next, if the new terms are inserted in their proper places, thus:

$$a \cdot a + \frac{b-a}{m+1} \cdot a + 2\frac{b-a}{m+1} \dots a + m \cdot \frac{b-a}{m+1} \cdot b \cdot b + \frac{c-b}{m+1} \cdot b + 2\frac{c-b}{m+1} \dots b + m \frac{c-b}{m+1} \cdot c \dots$$

each of the successive terms will differ from the succeeding term by $\frac{b-a}{m+1}$, and the whole succession, therefore, is a new arithmetical progression.

485. Corollary.—*The formula for inserting any number of arithmetical means between any two quantities is: $\delta' = \frac{b-a}{m+1}$, b and a being the given quantities, m the number of means to be inserted, and δ' the required common difference.*

486. II. Harmonical Progressions.

To insert the same number of harmonic means between the terms, two and two, of a harmonic progression.

Let $a, b, c, d, e, \dots$ be the given harmonic progression, and let m be the number of harmonical means to be inserted between a and b , b and c , etc. Taking the reciprocals of the terms, we obtain, $\frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c} \cdot \frac{1}{d} \dots$, an arithmetical progression of which the common difference is $\frac{a-b}{ab}$. Insert-

ing (484) m arithmetical means between $\frac{1}{a}$ and $\frac{1}{b}$, $\frac{1}{b}$ and $\frac{1}{c}$,

etc., we obtain,

$$\frac{1}{a} \dots \frac{b(m+1)+a-b}{ab(m+1)} \dots \frac{b(m+1)+2(a-b)}{ab(m+1)} \dots \frac{b(m+1)+m(a-b)}{ab(m+1)} \dots$$

$$\frac{1}{b} \dots \frac{c(m+1)+b-c}{bc(m+1)} \dots \frac{c(m+1)+2(b-c)}{bc(m+1)} \dots \frac{1}{c} \dots \dots \dots \text{Hence}$$

the new harmonical progression is $a, \frac{ab(m+1)}{b(m+1)+a-b}, \dots$
 $\frac{ab(m+1)}{b(m+1)+m(a-b)}, b, \frac{bc(m+1)}{c(m+1)+b-c}, \dots$
 $\frac{bc(m+1)}{c(m+1)+m(b-c)}, c.$

487.—III. Geometrical Progressions. To insert the same number of geometrical means between the terms, two and two, of any geometrical progression.

Let $\div a:b:c:d:e:f: \dots$ be the given progression, the ratio being ρ , and let m be the number of means to be inserted between a and b , between b and c , and so on. From definition, the means inserted between a and b must, when taken with a and b , form a minor geometrical progression of $m+2$ terms. Similarly those between b and c must, with b and c , form a progression, and so on. In the first of the minor progressions a is the first term, and b the $(m+2)$ th, and let ρ' denote the ratio. Then,

$$b = a\rho'^{m+2-1} = a\rho'^{m+1}; \text{ whence, } \rho' = \sqrt[m+1]{\frac{b}{a}},$$

and the first minor progression is,

$$\div a: a\sqrt[m+1]{\frac{b}{a}}: a\sqrt[m+1]{\frac{b^2}{a^2}}: \dots: a\sqrt[m+1]{\frac{b^m}{a^m}}: b.$$

Similarly the second minor progression is:

$$\div b: b\sqrt[m+1]{\frac{c}{b}}: \dots: b\sqrt[m+1]{\frac{c^m}{b^m}}: c,$$

and so on, the successive ratios, $\rho'', \rho''', \rho''''$, etc., being,

$$\sqrt[m+1]{\frac{c}{b}}, \sqrt[m+1]{\frac{d}{c}}, \sqrt[m+1]{\frac{e}{d}}, \text{ etc.}$$

These ratios are all equal, since $\frac{b}{a} = \frac{c}{b} = \frac{d}{c} = \frac{e}{d} = \dots$, etc.,

each fraction being equal to ρ . Hence, as the last term of each minor progression is the first term of the next, these progressions, placed in order, make up a new geometrical progression, thus:

$$a : a^{\frac{m+1}{m}} \sqrt[m]{\frac{b}{a}} : a^{\frac{m+1}{m}} \sqrt[m]{\frac{b^2}{a^2}} : \dots : a^{\frac{m+1}{m}} \sqrt[m]{\frac{b^m}{a^m}} : b : b^{\frac{m+1}{m}} \sqrt[m]{\frac{c}{b}} : b^{\frac{m+1}{m}} \sqrt[m]{\frac{c^2}{b^2}} : \dots : b^{\frac{m+1}{m}} \sqrt[m]{\frac{c^m}{b^m}} : c : \dots$$

$$\text{or, } a : \sqrt[m]{a^m b} : \sqrt[m]{a^{m-1} b^2} : \dots : \sqrt[m]{a b^{m-1}} : b : \sqrt[m]{a b^{m-1} c} : \sqrt[m]{b^m c} : \sqrt[m]{b^{m-1} c^2} : \dots$$

488. Corollary I.—*The formula for inserting any number of geometrical means between any two quantities is:*
 $\rho = \sqrt[m+1]{\frac{b}{a}}$; *a being the first, and b the second, of the given quantities, m the number of means to be inserted, and ρ the required ratio.*

489. Corollary II.—*The product of any two terms (of a geometrical progression) equally distant from the extremes, equals the product of the extremes, and the square of any geometrical mean equals the product of the adjacent extremes.*

490. In general, if the law of a series is known, and several terms are given, the interpolation may be effected by simply applying the law of the series.

If, again, a series be given of the form $a', b', c', \dots$, the quantities in which bear such a relation to some other quantities, $a, b, c, d, \dots$ that $a:b::a':b'$, $b:c::b':c'$, etc., it is clear that a, b, c, d, a', b', c' being given, and d' being required, we may find it by the proportion $c:d::c':x=d'$.

Whenever the terms in any series, as $a', b', c', \dots$ have a constant dependence (whatever its nature) on terms not in the series, as $a, b, c, \dots$ the former terms are called **Functions** of the latter, which are termed **Arguments**. Thus, if the terms

2	4	6	8
1.414	2.000	2.4495	2.8284

be given, the numbers in the lower line being the square roots of those in the upper, the latter, i. e., 2, 4, 6, 8, are called *arguments* and the former *functions*.

B. THE LAW OF THE SERIES IS NOT GIVEN.

491. If a series of functions be given, the law of the series being unknown, any term after the first may be found by simply applying formula (1, 481). n here represents the number of intervals from the first to the required term, the interval between the first and second being 1. Thus, if the series $\overbrace{a \quad b \quad c \quad d \quad e \quad e' \quad f}^{\text{---}}$, $a, b, c, \dots$ be given the $(n+1)th$ term is the one that has n intervals before it, the distance from a to b , or b to c , etc., being called 1 interval. If, now, a term is to be inserted between e and f , as at e' , $\frac{s}{q}$ of the distance from e to f , n will be greater by $\frac{s}{q}$ than the number of intervals preceding e ; thus, if there are r intervals between e and a , $n = r + \frac{2}{3} = \frac{3r+2}{3}$. In general:

If e' is $\frac{s}{q}$ of the way from e to f , and r intervals precede e , $n = r + \frac{s}{q}$ intervals, $= \frac{qr+s}{q}$. Replacing n , then, by $\frac{p}{q}$, it is clear that p is to be obtained by multiplying the number of entire intervals preceding the term to be inserted, by the denominator of the fraction indicating the distance of the required term from the next preceding, and to the product adding the numerator of this fraction; q is the denominator of the same fraction. The formula (481) then becomes

$$T = a + \frac{p}{q}D_1 + \frac{\frac{p}{q}(\frac{p}{q}-1)}{1.2}D_2 + \dots$$

When the series is such that all the terms of the rth order of differences are equal, i. e., is an arithmetical progression of the rth order, the formula, carried out as far as the $(r+1)th$ term, gives an exact result. When the series is not of this character, the result given by the formula is only approximate; and while the degree of closeness of the approximation depends on the number of terms taken, it is usually sufficient to take only the first order of differences. If greater accuracy is required, we may assume each term

of the first, or second, or third, etc., order of differences to be equal to the arithmetical average, thus making the first term of the next order taken 0. The average term of any order of differences is obtained by adding n terms together and dividing the result by n .

EXAMPLES.

$$\begin{array}{ll} 1. \text{ Given } \sqrt[3]{60}=3.914868, & D_1=.021629, \\ \sqrt[3]{61}=3.936497, & D_2=-.000235, \\ \sqrt[3]{62}=3.957891, & D_3=.000007, \\ \sqrt[3]{63}=3.979057, & \frac{p}{q}=\frac{33}{100}, \end{array}$$

$$\text{to find } \sqrt[3]{60.33},$$

$$a=3.914868$$

$$\frac{p}{q}D_1=0.007138$$

We neglect the third and following terms, as D_3 is practically 0.

$$\frac{\frac{p}{q}(\frac{p}{q}-1)}{2}D_2=0.000026$$

$$T=3.922032=\sqrt[3]{60.33}.$$

$$\begin{array}{ll} 2. \text{ Given } \sqrt{2}=1.4142136, & \text{find } \sqrt{5.76}. \\ \sqrt{4}=2.0000000, & \\ \sqrt{6}=2.4494897, & \\ \sqrt{8}=2.8284271, & \end{array}$$

3. A ship sailed from the equator; her position on a certain day, at noon, was $10^\circ 5' 38''.6$ N; on the next day, at noon, $13^\circ 6' 18''.86$ N; on the next, $16^\circ 6' 59''.41$ N; and on the fourth day, $19^\circ 7' 40''.24$ N. Required, the distance north at low 12 of the second day.

Here the times may be considered the arguments, and the distances the functions.

$$\begin{array}{ll} 1^\circ \text{ Order differences. } 3^\circ 40''.26 & 3^\circ 40''.55 & 3^\circ 40''.83. & \frac{s}{q}=\frac{1}{2}. \\ 2^\circ \text{ Order differences. } 0''.29 & 0''.28. & & \end{array}$$

4. Insert six arithmetical means between the terms, two and two, of the progression, $3\cdot 27\cdot 51\cdot 75\cdot 99\cdot \dots$

5. Insert four geometric means between the terms, two and two, of the progression, $\div 3 : 18 : 108 : 648 : \dots$

6. Which is the greatest, the Arithmetical, Harmonical, or Geometrical mean between two given quantities? Which is the least? *Ans.* A. M. and H. M., respectively.

7. The arithmetical mean, or average, of n quantities, being $\frac{1}{n}$ th of their sum, and the geometrical mean or average, being the n th root of their product, which of these two means is the greater?

SUMMATION OF SERIES.

I. Arithmetical Progressions.

492. Lemma.—*In any arithmetical progression, the sum of any two terms at equal distances from the extremes, equals the sum of the extremes.*

$a \cdot b \cdots c \cdots d \cdots e \cdots f \cdots \cdots j \cdots k \cdots l$, being any arithmetical progression, of which δ is the common difference, the n th term from the beginning is $a + (n-1)\delta$ (1). If, now, the order of the terms be reversed, thus, $l \cdots k \cdots j \cdots \cdots e \cdots d \cdots c \cdots b \cdots a$, the n th term from the beginning of the new progression will be the same as the n th term from the end of the old, and will be, since the common difference of the new progression (450) is $-\delta$, $l - (n-1)\delta$ (2). The sum of the values, (1) and (2), is $a + l$, the sum of the extremes.

493. Corollary.—*Any three quantities being in arithmetical progression, the middle term, i. e., the arithmetical mean, equals half the sum of the other two.*

494.—Theorem. *The sum of any number of terms of an arithmetical progression equals half the sum of the extremes, multiplied by the number of terms considered.*

Let $S = a + b + c + d + \cdots + j + k + l(1)$, a, b, c , etc., being in arithmetical progression. Reversing the terms in the second member of (1), $S = l + k + j + \cdots + d + c + b + a$ (2),

whence, adding (1) and (2), member to member,

$$2S = (a+l) + (b+k) + \cdots + (k+b) + (l+a),$$

Or, since there will be as many of the binomials $(a+l)$, $(b+k)$, etc., as there are terms, and (492), each equals $a+l$,

$$2S = n(a+l) \therefore S = \left(\frac{a+l}{2}\right)n.$$

This proposition may be expressed in the formula,

$S = \left(\frac{a+l}{2}\right)n$; and from this equation we may write at once the value of any one of the four quantities, in terms of the other three.

Here $\frac{a+l}{2}$ is often called the *mean term*, or *average term*, of the progression; it is the middle term if the number of terms is odd, and, when the number is even, expresses the value that each term would have, if, with the same sum, the terms were all equal. It is the *arithmetical mean* of the n terms of the progression; since $S = \left(\frac{a+l}{2}\right)n$, $\frac{a+l}{2} = \frac{S}{n}$. In other words, the arithmetical mean of any number of quantities is the sum of the quantities divided by the number of them.

495. From the formulas,

$$l = a + (n-1)\delta \quad (1), \text{ and } S = \left(\frac{a+l}{2}\right)n, \quad (2),$$

any two of the quantities, a , l , δ , n , S , may be found, the other three being given. For, any three being given, (1) and (2) become equations of the first degree, involving but two unknown quantities; and solving these equations (375), we have the quantities required. The different problems that may arise in this connection will be found, with their solutions, in Table I.

When, no three of these quantities being given directly, simply the number of terms is given with other conditions, and it is required to find the terms, the progression is generally represented by the use of two unknown quantities, in such a way, that in taking the sum of the terms, one of

these unknown quantities may disappear. If the number of terms be odd, let x represent the middle term, and y the common difference; then the progression will be,

$$\dots x - 2y \cdot x - y \cdot x + y \cdot x + 2y \dots$$

of the number of terms be even, let $x-y$ and $x+y$ represent the middle terms; $2y$ will be the common difference, and the progression will be,

$$\dots x - 7y \cdot x - 5y \cdot x - 3y \cdot x - y \cdot x + y \cdot x + 3y \cdot x + 5y \cdot x + 7y \dots$$

If, in the expression, $S = \left(\frac{a+l}{2}\right)n$, $n = \infty$, S will be the sum of an infinite number of terms, beginning at a , and will be $\pm \infty$, according as the progression is increasing or decreasing.

PROBLEMS.

1. What debt can be discharged in a year by paying \$2 the first week, \$5 the second, \$8 the third, and so on? What is the last payment?

2. Find three numbers in arithmetical progression, such that the sum of their squares shall be s , and the square of the mean shall be greater by d than the product of the extremes.

If $x-y \cdot x \cdot x+y$ represent the progression, we have, from the conditions, $(x-y)^2 + x^2 + (x+y)^2 = s$ or $3x^2 + 2y^2 = s$ (1), and $x^2 - y^2 = x^2 - d$ (2); whence $y = \pm \sqrt{d}$, and, substituting in (1), $3x^2 = s - 2d$ or $x = \pm \sqrt{\frac{s-2d}{3}}$.

3. In an arithmetical progression, the sum of four terms is 26, and the sum of the squares of these terms is 228. What are the terms?

Ans. $1 \cdot 5 \cdot 9 \cdot 11$.

4. A falling body reaches the earth in n seconds, descending a feet the first second, $3a$ the second, $5a$ the third, etc. From what height does it fall? How far does it fall during the last second?

Ans. an^2 ; $(2n-1)a$.

5. How many strokes does a common clock strike in the course of one day of 24 hours?

Ans. 156.

6. What is the n th term, and what is the sum, of the series $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \dots$? of the series $2 \cdot 4 \cdot 6 \cdot 8 \dots$?

Ans. $2n-1$; n^2 ; $2n$, $n+n^2$.

7. Insert 6 arithmetical means between the terms, two and two of the progression $2 \cdot 23 \cdot 44 \cdot 65 \dots$.

8. A, leaving R—, travels a miles the first day, $3a$ the second, $5a$ the third, etc. B, leaving the same place b days later, travels at a uniform rate of c miles per day. At what time will A and B be together?

Ans. $\frac{1}{2a}(c \pm \sqrt{c^2 - 4abc})$.

9. Find four numbers in arithmetical progression, such that the sum of the numbers shall be a , and the sum of their squares shall be c^2 .

II. Geometrical Progressions.

496. *The sum of any number of terms of a geometrical progression equals the quotient arising from dividing the product of the last term and ratio, diminished by the first term, by the ratio diminished by 1.*

Let $\begin{cases} \div a : b : c : d : \dots : k : l, & \text{or} \\ \div a : a\rho : a\rho^2 : a\rho^3 : a\rho^4 : \dots : a\rho^{n-2} : a\rho^{n-1}, \end{cases}$

be n terms of the given progression, ρ being the ratio, and denote their sum by S . Then,

$$S = a + a\rho + a\rho^2 + a\rho^3 + \dots + a\rho^{n-2} + a\rho^{n-1} \quad (1).$$

Multiplying both members by ρ ,

$$\rho S = a\rho + a\rho^2 + a\rho^3 + \dots + a\rho^{n-1} + a\rho^n \quad (2).$$

The terms of the second member of (1) after a , are the same as the terms of the second member of (2) before $a\rho^n$. Hence, if we subtract (1) from (2), member from member, these intermediate terms disappear, and we obtain,

$$\rho S - S = a\rho^n - a \quad \therefore S = \frac{a\rho^n - a}{\rho - 1};$$

Or, since $l = a\rho^{n-1}$, $S = \frac{l\rho - a}{\rho - 1}$, a formula in which the theorem (496) is expressed in algebraic language. From this

formula (an equation of the first degree, involving S , l , ρ , and a), the value of any one of them may be obtained when the other three are given.

497. To discuss the value of S , $S = \frac{l\rho - a}{\rho - 1} = \frac{a\rho^n - a}{\rho - 1}$.

I. When ρ is positive.

1°. $\rho > 1$. In this case, the progression is increasing, and (1) gives the value of S at once, both terms of the fraction being positive.

2°. $\rho < 1$. The progression is decreasing, and each term of the fraction, $\frac{l\rho - a}{\rho - 1}$, is negative, since l , and therefore $l\rho$, $< a$ and $\rho < 1$. Both terms being negative, the value of the fraction is positive, giving us a positive sum. But as it may often be inconvenient to deal with negative quantities in the terms, we write, in this case, $S = \frac{a - l\rho}{1 - \rho}$, in which both terms are positive. This value is obtained either by multiplying both terms of the former fraction by -1 , or by subtracting (2) from (1), Article 496, and then solving with reference to S .

3°. $\rho = 1$. In this case, $S = \frac{0}{0}$; but this is a vanishing fraction, which, evaluated, gives

$$a \frac{(\rho^n - 1)}{\rho - 1} = a(\rho^{n-1} + \rho^{n-2} + \rho^{n-3} + \dots + 1);$$

placing $\rho = 1$, the parenthesis becomes $1 + 1 + 1 + \dots$ to n terms, *i. e.*, n . Hence, $S = an$. This is evidently as it should be, for, under the supposition, every term of the progression is equal to the first, and therefore the sum of n of them is n times the first term.

II. If $\rho = 0$, every term after the first becomes 0, and S reduces to a , as it should.

III. When ρ is negative.

Let it equal $-\rho'$. Then $S = \frac{-l\rho' - a}{-\rho' - 1} = \frac{l\rho' + a}{\rho' + 1}$. If, when a is positive, n is even, l is negative; if n is odd, l is posi-

tive. Then $S = \frac{a-l\rho'}{1+\rho'}$, in the first case, and $\frac{a+l\rho'}{1+\rho'}$ in the second, taking for l its numerical value. The sum will then be positive or negative in the first case, according as the progression is decreasing or increasing. In a decreasing progression then, of which the first term is positive, and the succeeding terms alternately negative and positive, the sum of any number of terms is positive. In an increasing progression, whose terms are alternately positive and negative, (the first term being positive), the sum of an even number of terms is negative, and the sum of an odd number is positive.

If a is negative, $S = \frac{-a+l\rho'}{1+\rho'}$ or $-\frac{a+l\rho'}{1+\rho'}$, according as n is even or odd. From these we see that in a decreasing progression, beginning with a negative term, the succeeding terms being alternately positive and negative, the sum of the terms is negative. If the progression is increasing, the sum of the terms is positive or negative, according as the number considered is even or odd.

498. To find the sum of an Infinite number of terms of a decreasing geometrical progression.

The sum of the terms of a decreasing progression is $\frac{a-l\rho}{1-\rho} = \frac{a-a\rho^n}{1-\rho}$, in which ρ is a fraction, <1 . The higher the power to which we raise a fraction <1 , the smaller will be the result. Hence, for large values of n , ρ^n will be small in comparison with the other quantities. When n becomes very large, ρ^n becomes very small; and finally, when n becomes greater than any assignable quantity, ρ^n becomes less than any assignable quantity or zero, in which case $a.\rho^n = a.0 = 0 = l\rho$; and, therefore, when

$$n = \infty, a-l\rho = a-a\rho^n = a-0 = a, \text{ and } S = \frac{a}{1-\rho}.$$

Hence, the sum of an infinite number of terms of a decreasing geometrical progression equals the first term divided by 1 minus the ratio.

$\frac{a}{1-\rho}$ may be regarded as the limit, which the sum of n terms approaches as n increases. The error occasioned by taking $\frac{a}{1-\rho}$ for the sum of n terms is $\frac{a\rho^n}{1-\rho}$, which gradually diminishes as n increases, until, when $n=\infty$, the error is 0.

The formula $S = \frac{a}{1-\rho}$ is very important; we are not often able to express, in finite terms, the sum of an infinite series.

PROBLEMS.

1. Find the sum of 15 terms of the progression
 $\div 3:6:12:24:\dots$

2. Find the sum of 11 terms of the progression
 $\div 3:9:27:\dots$ *Ans.* 265719.

3. Insert 7 geometrical means between 2 and 128.

4. Find four numbers in geometrical progression, such that the first, minus the third, shall equal a , and the second, minus the fourth, shall equal b .

Let $x =$ the first term and $y =$ the ratio. The progression, then, is $\div x:xy:xy^2:xy^3$; whence, $x - xy^3 = a$, and $xy - xy^4 = b$; or, $x(1 - y^3) = a$ (1), and $xy(1 - y^3) = b$ (2) $\therefore y = \frac{b}{a}$, and substituting in (1), $x = \frac{a^3}{a^3 - b^3}$.

5. Find six numbers in geometrical progression, such that the sum shall be 1365, and the sum of the third and fourth shall be 80.

x being the first term, and y the ratio, we have

$$x + xy + xy^2 + xy^3 + xy^4 + xy^5 = 1365 \quad (1)$$

$$\frac{xy^2 + xy^3}{x + xy} = 80 \quad (2), \text{ whence,}$$

$$x + xy + xy^4 + xy^5 = 1285 \quad (3), \text{ or,}$$

$$x(1 + y) + xy^4(1 + y) = 1285, \text{ or}$$

$$(1 + y^4)(1 + y)x = 1285 \quad (4).$$

Writing (2), $xy^2(1 + y) = 80$, and dividing, member by mem-

ber, by (4), we obtain $\frac{y^2}{1+y^4} = \frac{80}{1285}$ or $y^4 - \frac{1285}{80}y^2 = -1 \therefore y=4$, and $x = \frac{80}{y^2+y^4} = 1$. The numbers, then, are 1, 4, 16, 64, 256 and 1024.

6. Suppose a body to move as follows, viz.: 30 miles the first second, 29 the second, $28\frac{1}{30}$ the third, and so on. If the body move in this way for eternity, how far will it travel altogether? *Ans.* 900 miles.

7. Find the value of .888 circulating to infinity. *Ans.* $\frac{8}{9}$.
Of .32. Of .2041. *Ans. to last,* $\frac{2021}{9900}$.

8. Four men receive \$700, the shares being in geometrical progression; the difference between the greatest and least is to the difference between the other two as 37 is to 12. Required, the respective shares.
Ans. \$108, \$144, \$192, \$256.

III. Recurring Series.

499. To find the Sum of a Recurring Series.

1°. If the series is of the first order, $S = \frac{a\rho^n - a}{\rho - 1}$ is the expression for the sum of n terms, ρ being the scale (450). In an increasing series, if $n = \infty$, $S = \infty$; in a decreasing series, if $n = \infty$, $S = \frac{a}{1-\rho}$.

2°. If the series is of the second order, $S = \frac{a+b-as''}{1-s'-s''}$ is the expression for the sum of an infinite number of terms, s' , s'' being the scale. For, $a, b, c, d, \dots$ being the terms, we have:

$$a=a$$

$$b=b$$

$$c=s'a+s''b$$

$$d=s'b+s''c$$

$$e=s'c+s''d$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

Adding these equations, member to member, and substituting for

$a+b+c+d+\dots$ the symbol S , we obtain $S=a+b+(S-a)s'+Ss''$;

whence, $S = \frac{a+b-as''}{1-s'-s''}$. If, now, the sum of a finite number (n) of terms be

required, denote by S_n the sum of the n terms.

Then $a=a$ If the terms after t extend to infinity,
 $b=b$ $u+v+w+\dots$ their sum, as just shown,
 $c=s'a+s''b$ is $\frac{u+v-us''}{1-s'-s''}$. Supposing $a, b, c, \dots$
 $d=s'b+s''c$ to extend to infinity, we have, for their
 $\dots\dots\dots$ sum $\frac{a+b-as''}{1-s'-s''}$. Hence, the sum of the
 $t=s'r+s''q$

finite portion $a\dots t$ of the given series, is

$$\frac{a+b-as''}{1-s'-s''} - \frac{u+v-us''}{1-s'-s''}, \text{ or } S_n = \frac{a+b-(u+v)-(a-u)s''}{1-s'-s''}.$$

3°. If the series be of the third order,

$S = \frac{a+b+c-(a+b)s'''-as''}{1-s'-s''-s'''}$ is the expression for the
 sum of an infinite number of terms, s''' , s'' , s' , being the
 scale.

For $a=a$

$$b=b$$

$$c=c$$

$$d=as'+bs''+cs'''$$

$$e=bs'+cs''+ds'''$$

$$\dots\dots\dots$$

$$\dots\dots\dots$$

Adding these equations, member
 to member, and placing, as be-
 fore, $S=a+b+c+d+\dots$ to in-
 finity, we obtain $S=a+b+c+Ss'$
 $+(S-a)s''+(S-a-b)s'''$; whence,
 $S = \frac{a+b+c-(a+b)s'''-as''}{1-s'-s''-s'''}$.

The sum of a finite number of terms may be found in a
 manner similar to that illustrated in 2°.

4°. The general method here employed is evidently
 applicable to any recurring series whatever. Hence, if t
 be the n th term of a recurring series of the n th order, and
 t' the $(n-1)$ th term,

$$S = \frac{a+b+c+\dots+t-(a+b+\dots+t')s''-\dots-(a+b)s''-as''}{1-s'-s''-s'''-\dots-s^{(n)'}}$$

will be the expression for the sum of an infinite number of
 terms of the recurring series, s'' , $\dots$ s''' , s'' , s' , being the
 scale.

The sum of a finite number of terms of a series of the
 n th order may be found by the general method exemplified
 in 2°.

If the series be arranged according to the powers of an indeterminate, the expressions in $1^0, 2^0, 3^0, 4^0$, enable us to determine the generating fraction, i. e., the fraction from which the series is developed. To take a single example, let the series be

$$1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7+\dots$$

The scale (478) is $-x^3, x^3, x$, and the sum $\frac{1}{1-x-x^3+x^5}$, from which fraction the series may evidently be developed.

500.—IV. Decreasing Series, in which the terms are alternately positive and negative.

In a decreasing series, in which the terms are alternately positive and negative, the sum of n terms is the sum of the series, to within less than the numerical value of the following term.

a, b, c, d , etc., being positive quantities, and $a > b > c > d > e$, etc., let $a-b+c-d+e-f+g-$, etc., represent any series of the kind mentioned in the proposition. The series is converging, since its sum is positive and $< a$. For, to obtain the sum, S , we must add to a the expressions $(-b+c)$, $(-d+e)$, $(-f+g)$, etc.; since $b > c$, $d > e$, $f > g$, etc., these quantities are all negative, whence their sum is negative, and we must add to a a negative quantity to get S . $\therefore S < a$. Again, $S > a-b$, for we must add to $a-b$ the series of positive quantities $(c-d)$, $(e-f)$, $\dots$ to get S . S , then, is positive, since it is greater than the positive quantity $a-b$, and S being positive and $< a$, the series is converging.

Now, suppose g is the n th term of the series. To get the sum of the series, we must add to $a-b+c-d+e-f+g$ the succession of negative quantities, $(-h+i)$, $(-j+k)$, $(-l+r)$, $\dots$, $h, i, j, \dots$, being the following terms of the series; $(-h+i)$ is negative, since $h > i$, $-j+k$ is negative, since $j > k$, and so on). In order to get S , then, having the sum of the first n terms, we must add to this sum, which we shall call S_n , the sum of a certain number of negative quantities. As the sum of these negative quantities must be negative, we must diminish S_n in order to get S , i. e., $S < S_n$.

Now, consider the first $n+1$ terms, the sum of which, $a-b+c-\dots-f+g-h$, denote by S_{n+1} . To obtain S from S_{n+1} , there must be added to the latter the sum of all the terms after h , i. e., $(i-j)+(k-l)+(r-s)+\dots$. Since $i > j$, $i-j$ is positive; since $k > l$, $k-l$ is positive, and so on; every couplet to be added to S_{n+1} to obtain S , is positive, The sum of all these positive quantities being itself positive, S_{n+1} must be *increased* by this sum, to produce S ; hence $S > S_{n+1}$. Since $S < S_n$ and $S > S_{n+1}$, S must lie between S_n and S_{n+1} , and must, therefore, differ from either by a less quantity than that indicating the difference between them; (for example, if a number is between 6 and 7, it must differ from either by a fraction, less than 1). Now $S_n - S_{n+1} = h$; hence $S - S_n < h$ and $S - S_{n+1} < h$. The error committed by taking $(S_{n+1}$ or) S_n for S is, therefore, numerically less than h , i. e., *The error committed by taking the sum of any number of terms for the sum of the series, is numerically less than the following term:*

If a series is of the form $a' + b' + c' + d' + a - b + c - d + \dots$, the quantities a' , b' , c' , etc., succeeding each other in any manner whatever, the preceding proposition applies to the part of the series beginning at a ; the sum of a' , b' , c' , etc., must be obtained independently, but, if a' , b' , etc., are finite, the sum of the series is always less than $a' + b' + \dots + d' + a$.

501.—V. Series derived from the expression $\frac{q}{k(k+p)}$.

Summation by Auxiliary series.

Let there be given a series the terms of which can be derived from the expression $\frac{q}{k(k+p)}$, by giving to p a fixed value, and attributing suitable values to q and k , and two auxiliary series formed from $\frac{q}{k}$ and $\frac{q}{k+p}$, so that the values of p , q and k , in any term, shall be the same as in the corresponding terms of the given series. The series derived

from $\frac{q}{k(k+p)}$ is the *primitive series*; the one derived from $\frac{q}{k}$ is called the *first auxiliary series*; the one derived from $\frac{q}{k+p}$ is called the *second auxiliary series*.

Lemma.—Any term of the primitive series equals $\frac{1}{p}$ multiplied by the excess of the corresponding term in the first auxiliary series over the corresponding term in the second.

From the nature of the case, $\frac{q}{k(k+p)}$ may represent any term of the primitive series, $\frac{q}{k}$ and $\frac{q}{k+p}$ representing the corresponding terms of the auxiliary series.

$\frac{q}{k} - \frac{q}{k+p} = \frac{kq + pq - kq}{k(k+p)} = \frac{pq}{k(k+p)}$, and multiplying this by $\frac{1}{p}$, we obtain $\frac{q}{k(k+p)}$.

502. Sum of the Series.—The sum of any number of terms in the primitive series equals $\frac{1}{p}$ multiplied by the excess of the sum of the corresponding terms in the first auxiliary series over the sum of the corresponding terms in the second.

This is a direct consequence of the lemma, for, that proposition being true of each term, this must be true of the sum of the terms.

Or, we may arrive at the result in a slightly different way. Let the series be,

Primitive, $a \ b \ c \ d \ e \dots \dots \dots t \ s \ x \ (1).$

1^o Auxiliary, $a' \ b' \ c' \ d' \ e' \dots \dots \dots t' \ s' \ x' \ (2).$

2^o Auxiliary, $a'' \ b'' \ c'' \ d'' \ e'' \dots \dots \dots t'' \ s'' \ x'' \ (3).$

Then, by (501), $a = \frac{a' - a''}{p}$, $b = \frac{b' - b''}{p}$, $c = \frac{c' - c''}{p}$, etc., or, substituting in (1), and indicating the addition,

$$S = \frac{a' - a''}{p} + \frac{b' - b''}{p} + \frac{c' - c''}{p} + \dots + \frac{s' - s''}{p} + \frac{x' - x''}{p}$$

$$= \frac{a' + b' + c' + d' + \dots + s' + x' - (a'' + b'' + c'' + \dots + s'' + x'')}{p}$$

The practicability of this method of summing series depends on the fact that, in the subtraction, each of the series $a' + b' + c' + \dots$, and $a'' + b'' + c'' + \dots$, destroys all but a finite portion of the other, which finite portions we are able to sum, by actual addition, or otherwise. Whenever we can find the difference between the auxiliary series, we may sum the primitive series, even though the number of terms be *infinite*.

The *general method* here indicated may be extended to series derived from expressions of the form, $\frac{q}{k(k+p)(k+2p)} \dots \frac{q}{k(k+p) \dots (k+np)}$, and other fractions, but the discussion of these more general cases is beyond the scope and limits of an elementary work.

503. VI.—Series of which the law may, or may not be known; **Series in General.** The most general method of summing series is the **Method by Differences**, or **Differential Method**.

Let $a, b, c, d, \dots$ (1) represent any series whatever, the sum of n terms of which is required. Assume the series $0, a, a+b, a+b+c, a+b+c+d, \dots$ (2), the first order of differences in which is $a, b, c, d, \dots$ the same as the given series. A comparison of (2) and (1) shows that the third term of (2) is the sum of two terms of (1), the fourth term of (2) the sum of three terms of (1), and in general, the n th term of (2) is the sum of $n-1$ terms of (1). Indeed, this is a direct consequence of assuming (2) so that its first order of differences should coincide with (1); for, as the terms in the first order of differences are the differences between consecutive terms of (2), any term of (2) may be obtained by adding to the preceding the term of (1) corresponding to this preceding. Thus, beginning with 0, the second term of (2) is $0+a$, or a ; the third term, $a+b$; the fifth term, $a+b+c+d$; and the $(n+1)$ th term the sum of the first n terms of (1). Hence, if the $(n+1)$ th term of (2) is obtained

in any way, the sum of the first n terms of (1) is obtained. Now the most general method of finding any term of a series, the only method applicable to all series, is the method by differences, the formula of which is

$$T = a + nD_1 + n\frac{n-1}{2}D_2 + \dots (3)$$

To apply this formula to (2), we must substitute for a the first term of (2), and for $D_1, D_2, \dots$ the first terms of the successive orders of differences. As the first order of differences of (2) is (1), the second order of differences of (2) is the same as the first order of differences of (1), and in general, the first term of any order of differences of (2) will be the same as the first term of the next lower order of (1). Hence the first terms of the first, second, third, etc., orders of differences of (2) will be $a, D_1, D_2, \dots$. Making, in (3), the substitutions here indicated, we obtain, for the $(n+1)th$ term of (2), $T = 0 + na + \frac{n(n-1)}{1.2}D_1 + n\frac{(n-1)(n-2)}{1.2.3}D_2 + \dots$

This, however, is the sum of the first n terms of (1). Hence, indicating this sum by S_n , we obtain,

$$S_n = na + \frac{n(n-1)}{1.2}D_1 + \frac{n(n-1)(n-2)}{1.2.3}D_2 + \dots + D_{n-1} (4).$$

504. If all the terms of any order of differences are equal, the terms of the succeeding orders reduce to 0, and (4) gives the exact sum of n terms of the given series; all terms after that containing D_e reducing to 0 if the terms of the e th order of differences are all equal, in which case the series is an arithmetical progression of the e th order. If the terms of no order are all equal, the formula must be carried out to n terms (to D_{n-1}), to ensure an exact result, since only in this case will (3) give T exactly. In this event n terms of the series (1) must be given (480), and the formula is useless as far as exact results are concerned, for it is much simpler, and certainly more direct, to add the given n terms of the series at once than to deduce $n-1$ orders of differences, substitute in (4), and then add n

terms of it, except in the very unlikely event of (4) being a progression, or other simple series, which can be summed much more easily than (1). If an exact result is not required, the formula is of service, for an approximate result is obtained by taking the sum of any number of terms (less than n) of the second member of (4). The degree of closeness of the approximation is necessarily determined by the number of terms taken, and the relative numerical value of the terms; if, in (4), the terms increase toward the right, or diminish very slowly in this direction, the approximation, if any of the later terms be neglected, will be very coarse, while, if the terms of (4) diminish rapidly toward the right, the approximation will be sufficiently close for all practical purposes. If the term involving D_c is the last one taken, the accuracy of the work may be increased by substituting for D_c the *arithmetic mean*, or *average*, of the terms of the c th order of differences. The number of terms of (4) to be taken, in any case in which an approximate result is desired, must, therefore, be determined by the nature of the series.

The formula gives exactly the sum of an arithmetical progression of any order whatever, the number of terms of the second member being greater by 1 than the number indicating the order of the progression. Thus, in a progression of the first order, two terms must be taken, in one of the second order three, and so on; the number taken in a progression of the r th order being $r+1$, since all terms after that involving D_r reduce to 0. If the progression be of the first order, all the terms of the first order of differences will be equal, and $S_n = na + n \frac{n-1}{2} D_1$, or since $D_1 = \delta$,
$$S = n \left(a + \frac{n-1}{2} \delta \right) = n \frac{2a + (n-1)\delta}{1.2} = n \frac{a + (a + (n-1)\delta)}{2} = n \frac{a + l}{2},$$
 the same expression previously deduced.

When $n = \infty$, the second member of (4) becomes $\pm \infty$. This result we should anticipate, for, in the case of an arith-

metrical progression, $\pm\infty$ is the true sum, positive or negative, as the series is increasing or decreasing. In any other case, the infinite result is an indication of the failure of the formula, a failure to be expected, since the number of terms essential to an exact result is, except in the cases already noted, $n+1$, which, if $n=\infty$, is ∞ .

505. VII.—If none of the methods now indicated be expedient in any particular case, the only general method of summing a series is the method of direct addition of the terms. If the given series be finite in the number and value of its terms, the sum may be obtained exactly; but if the series be infinite, the sum can be obtained only approximately, the degree of closeness of the approximation depending on the nature of the series (as to convergence, etc.), and the number of terms taken.

506. Of the methods of summing series, all that give exact results, except VII., are applicable when the number of terms to be summed is infinite, and I-V are never used except when they give exact results. While the special nature and consequent limited applicability of these methods is a serious inconvenience, the fact of their giving the sum of an infinite number of terms is a great advantage. The method by differences, or differential method, on the other hand, is often used even when its use leads to merely approximate results, and is theoretically applicable to any series whatever. Its importance depends largely on this extended applicability, though its practical utility is impaired, in different degrees, by the various defects already pointed out.

EXAMPLES.

Find the sum of the following series:

1. $1 \pm x + x^2 \pm x^3 + x^4 \pm x^5 + \dots$

2. $x + 3x^2 + 6x^3 + 10x^4 + \dots$

3. $1 + 3x + 5x^2 + 7x^3 + \dots$

Ans. $\frac{x}{(1-x)^2}$

Ans. $\frac{1+x}{(1-x)^2}$

- 4.
- $x+2x^2+3x^3+4x^4+\dots$
- ; when the series is decreasing.

$$\text{Place } S=x+2x^2+3x^3+4x^4+\dots$$

$$-2xS= -2x^2-4x^3-6x^4-\dots$$

$$x^2S= \quad \quad +x^2+2x^3+\dots$$

$$(1-2x+x^2)S=x \therefore S=\frac{x}{(1-x)^2}$$

5. $\frac{a}{b}-\frac{ac}{b^2}x+\frac{ac^2}{b^3}x^2-\frac{ac^3}{b^4}x^3+\dots$

Ans. $\frac{a}{b+cx}$

6. $2-6+18-54+162-\dots$

7. $54-53+52-51+50-\dots$ to 20 terms.

8. $60-59+58-\frac{1}{60}-57-\frac{179}{3600}+\dots$ to 10 terms.

9. $1-\frac{2}{6}+\frac{3}{20}-\frac{4}{56}+\frac{5}{144}-\frac{6}{352}+\dots$ to 10 terms.

10. $\frac{2}{3.5}-\frac{3}{5.7}+\frac{4}{7.9}-\frac{5}{9.11}+\frac{6}{11.13}-\dots$

This may be derived from $\frac{q}{k(k+p)}$ by placing $p=2$,
 $q=2, -3, +4, -5, +6, \dots$ and $k=3, 5, 7, 9, \dots$

1°. Auxiliary series,

$$\frac{2}{3}-\frac{3}{5}+\frac{4}{7}-\frac{5}{9}+\frac{6}{11}-\dots \mp \frac{n+1}{2n+1}$$

2°. do. $\frac{2}{5}-\frac{3}{7}+\frac{4}{9}-\frac{5}{11}+\dots \pm \frac{n}{2n+1} \mp \frac{n+1}{2n+3}$

Notice that numerators and denominators form arithmetical progressions, the n th terms of which find by (474); or find the n th terms by induction from the law of formation of any of the first terms.

Then $S=\frac{1}{2} \left[\left(\frac{2}{3} \pm \frac{n+1}{2n+3} \right) - (1-1+1-1+\dots \pm 1) \right]$,
 the upper or lower sign being taken, as n is even or odd.
 When n is even, $S=\frac{1}{2} \left[\left(\frac{2}{3} + \frac{n+1}{2n+3} \right) - 1 \right]$, as, there being $n-1$ terms in the second parenthesis, this reduces to

1 when n is even. Hence

$$S = \frac{1}{2} \left(-\frac{1}{3} + \frac{n+1}{2n+3} \right) \quad \text{If } n \text{ is odd; } S = \frac{1}{2} \left(\frac{2}{3} - \frac{n+1}{2n+3} \right)$$

When $n = \infty$, $\frac{n+1}{2n+3}$ apparently reduces to $\frac{\infty}{\infty}$, a symbol

of indetermination. But $\frac{n+1}{2n+3} = \frac{1 + \frac{1}{n}}{2 + \frac{3}{n}}$, which reduces to

$$\frac{1}{2} \text{ when } n = \infty. \text{ Hence, } S_{n=\infty} = \frac{1}{2} \cdot \frac{1}{6} = \frac{1}{12}.$$

$$11. \frac{1}{1.3} + \frac{1}{2.4} + \frac{1}{3.5} + \frac{1}{4.6} + \dots \quad \text{Ans. } S_{n=\infty} = \frac{3}{4} \text{ or } \frac{1}{4}.$$

$$12. \frac{1}{1.5} + \frac{1}{5.9} + \frac{1}{9.13} + \frac{1}{13.17} + \dots \quad \text{Ans. } \frac{1}{4}.$$

$$13. \frac{1}{3} + \frac{1}{9} + \frac{1}{18} + \frac{1}{30} + \dots$$

This may be placed under the form

$$\frac{2}{3} \left(\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5} + \dots \right) \quad \text{Ans. } S_{n=\infty} = \frac{2}{3}.$$

$$14. \frac{1}{3.8} + \frac{1}{6.12} + \frac{1}{9.16} + \dots \quad \text{Ans. } S_{n=\infty} = \frac{1}{12}.$$

Find the sum of n terms of the following series:

$$15. 1, 1+2, 1+2+3, 1+2+3+4, \dots$$

$$a=1, D_1=2, D_2=1, D_3=0, D_4=0, \text{ etc.}$$

$$S = n + \frac{n(n-1)}{1 \cdot 2} \cdot 2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}.$$

$$16. 6, 24, 60, 120, \text{ etc.}$$

$$a=6, D_1=18, D_2=18, D_3=6, D_4=0, D_5=0, \text{ etc.}$$

$$S = \frac{n(n+1)(n+2)(n+3)}{4}.$$

$$17. 1, 4, 9, 16, 25, 36, \dots \quad \text{Ans. } \frac{n(n+1)(2n+1)}{6}.$$

$$18. (m+1), 2(m+2), 3(m+3), 4(m+4), \dots$$

$$\text{Ans. } \frac{n(n+1)(1+2n+3m)}{6}.$$

$$19. 1, 4, 10, 20, 35, \dots \quad \text{Ans. } \frac{n(n+1)(n+2)(n+3)}{1 \cdot 2 \cdot 3 \cdot 4}$$

$$20. 1, 4, 8, 13, 19, \dots \quad \text{Ans. } \frac{n(n^2+6n-1)}{6}$$

$$21. 1, 2^4, 3^4, 4^4, 5^4, \dots \quad \text{Ans. } n\left(\frac{n^4}{5} + \frac{n^3}{2} + \frac{n^2}{3} - \frac{1}{30}\right)$$

$$22. 70, 252, 594, 1144, \dots \quad \text{Ans. } \frac{n(6n^3+44n^2+99n+61)}{3}$$

$$23. 3, 10, 28, 62, 117, 198, \dots$$

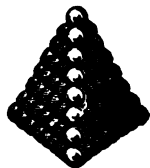
$$24. 10, 18, 29, 50, 90, 160, 273, 444, \dots$$

$$25. 17, 27, 47, 85, 152, 262, 432, 682, \dots$$

PILING BALLS AND SHELLS.

507. In navy yards, arsenals, etc., cannon balls and shells are usually stored on a level surface, in piles of three shapes, viz.: *triangular*, *square*, and *oblong*.

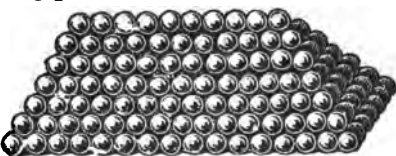
I. The triangular pile is composed of successive layers, each in the shape of an equilateral triangle, the number of balls in each side of the layers increasing from the top, which contains 1, to the lowest layer, which has as many balls on each side as there are layers. Hence, the number of balls in a complete triangular pile is the sum of the series, $1, 1+2, 1+2+3, \dots, 1+2+3+\dots+n$, n being the number of courses. The sum of this series, and, therefore, the number of balls in a complete triangular pile is, (Ex. 15, page 435), $S = \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3}$.



II. The complete square pile is made up of successive layers, square in shape, the number of balls in each side of the squares increasing from the top, where it is 1, to the lowest layer, where it is n , the number of layers. Hence, in a complete square pile the number of balls is $1^2+2^2+3^2+4^2+\dots+n^2$, or (Ex. 17, page 435), $S = \frac{n(n+1)(2n+1)}{1 \cdot 2 \cdot 3}$.



III. The complete oblong pile is made up of layers, rectangular in shape; the second layer from the top may be considered as made up of two lines of balls, each line con-



taining one ball more than the top row; the third layer consists of three lines, each having one ball more than either line of the second layer, and so on. In the long face of the pile, then, the number of balls in the successive lines increases by one in each line, from the top downward, and in the n th line, as the number of balls has increased by one for each line, the number of balls will be greater by $n-1$ than the number in the top row. In the end of the pile, the number of balls in any line will be the number indicating the distance from the top, and in the n th line will be n . Hence, if $t+1$ represent the number of balls in the top row, the number in the complete oblong pile will be the sum of the series $(t+1), 2(t+2), 3(t+3) \dots n(t+n)$, or (Ex. 18, page 435), $S = \frac{n(n+1)(1+2n+3t)}{1 \cdot 2 \cdot 3}$.

If any of these piles be incomplete, the number of balls that would be in the pile if it were complete, should be computed first; this, minus the number required to complete it, is the number actually in the pile. Or, suppose as many terms to be wanting from the beginning of the corresponding series, as there are courses wanting in the given pile; then sum the remaining terms of the series.

508. The preceding formulæ may be written:

$$S = \frac{1}{3} \frac{n(n+1)}{2} (n+1+1); \quad S = \frac{1}{3} \frac{n(n+1)}{2} (n+n+1);$$

$$S = \frac{1}{3} \frac{n(n+1)}{2} (\overline{i+n} + \overline{i+n} + \overline{i+1}).$$

Now, the number of balls in the triangular face of each of the piles is the sum of n terms of the arithmetical progression, $1 \cdot 2 \cdot 3 \dots$ or (494) $\left(\frac{1+n}{2}\right)n$. Hence, $\frac{n(n+1)}{2}$

expresses the number of balls in the triangular face of each pile. The first term of the second factor, in each case, is the number in the longest side of the base; the second term, the number in the opposite side of the base, and the third, the number in the top row. The number of balls in a pile of any one of the shapes mentioned may, therefore, be obtained by the following

RULE.—*Add to the number of balls in the longest side of the base the number in the opposite side thereof, and the number in the top parallel row. Multiply this sum by one third the number in the triangular face of the pile.*

EXAMPLES.

1. Required, the number of balls in a triangular pile of 20 courses; of 40 courses; of 50 courses.

Ans. 1540; 11,480; 22,100.

2. Required, the number of balls in a complete square pile of 30 courses; of 35 courses; of 40 courses.

Ans. 9455; 14,910; 22,140.

3. Required, the number of balls in the last course of each of the piles mentioned.

4. Required, the number of balls in a complete oblong pile, the number in the top row being 41, and the numbers in the sides of the lowest course being 70 and 30, respectively.

Ans. 28,055.

5. Required, the number of balls in an incomplete triangular pile, the number in each side of the top course being 7, and in each side of the bottom 40.

6. Required, the number of balls in an incomplete square pile, the number in each side of the upper course being 13, and in each side of the lowest course 50.

7. Required, the number of balls in an incomplete oblong pile, the numbers in the upper courses being 81 and 20, and in the lower courses 106 and 45.

CHAPTER XI.

LOGARITHMS.

509. The Logarithm of a number is the exponent of the power to which it is necessary to raise a fixed number in order to produce the given number.

Thus, in the equation $a^x=N$, a being a fixed number, and N any given number, x , the exponent of the power to which a must be raised, that N may result, is the *logarithm* of N . If, for a fixed value of a , N be supposed to have every possible numerical value, the resulting values of x form what is called a *System* of logarithms, and the fixed value of a is called the *base* of the system. If a second value be given to a , we have a second system of logarithms, and in general there is a system of logarithms for every value that can be given to a . Now, any positive value, except 1, may be assigned to a . No negative values can be assigned, for the even fractional powers of negative quantities are imaginary, and the integral powers are positive or negative, as the exponent is even or odd. If, then, a negative value were assigned to a , there would be no numbers corresponding to even fractional values of x , and no positive numbers corresponding to odd values of x . Again, 1 cannot be taken as the base of a system, for, every power of 1 being 1, no number except 1 would have a logarithm in such a system, and the logarithm of 1 itself would be indeterminate. As we can assign to a an infinite number of positive values besides 1, an infinite number of systems of logarithms is possible.

510. Even fractional powers of any positive number have the double sign; odd fractional powers are positive. If, therefore, the logarithms of negative numbers were considered at all, these numbers would fall in two classes:

those having logarithms, and those not having logarithms. To avoid the confusion that would inevitably result if we were called upon to discriminate in the case of every negative number met with, we discard altogether the consideration of the logarithms of negative numbers. This is often, but inaccurately, expressed thus: "Negative numbers have no logarithms." The notation $\log_a N$, etc., is often used to indicate the system in which the logarithm of a given number is to be taken, the subscript indicating the base of the system.

511. Three systems are in common use. The first was invented by **Napier** (Baron of Merchiston, Scotland), in 1614. Though often called, particularly by European writers, the **Hyperbolic** system, its relation to the rectangular hyperbola is not essentially different from that of the other systems to the common hyperbola. Its base is 2.71828 18284 59045 23536 02874+.

The **Natural System** is used in all theoretical discussions in the Higher Mathematics. Its base is the number e , equal to the sum

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \dots$$

This number agrees very nearly with the **Napierian base**, as already given, and **Natural** and **Napierian** logarithms were long treated as identical. They are not so, however, the relation between the two systems being thus expressed:

$$e^l = 10^L e^{-\frac{L}{10}}, \text{ or } L = 10^L \text{ nat. log. } (10^L) - 10^L,$$

in which equations L represents the natural logarithm of any number, l the Napierian logarithm of the same number, and e the natural base. The Napierian logarithms are the only ones, however, aside from the common logarithms, for which tables have been computed; and hence

in comparing with the common system we shall use the Napierian instead of the Natural logarithms. This will not affect the numerical exactness of the propositions, so far as we have to deal with them, but the student should remember that theoretical accuracy would require us to take note of the distinction just indicated.

The **Common, Denary, or Briggean** system has 10 as its base. It was published by Briggs, a Professor of Mathematics in Oxford, in 1617, after consultation with Napier. It is now used to the exclusion of all others, in actual numerical calculations.

Common logarithms will be denoted usually by the abbreviation $\log$, Natural logarithms by $\ln$, and Napierian logarithms by $\log_e$.

GENERAL PROPERTIES OF LOGARITHMS.

512. I. *The logarithm of any product is equal to the sum of the logarithms of the factors thereof.*

Let N' and N'' be two numbers, the logarithms of which, in a system with any base whatever, as a , are x' and x'' . Then, (509), $a^{x'} = N'$ (1), $a^{x''} = N''$ (2).

Multiply (1) by (2), member by member.

II. *The logarithm of any quotient is equal to the logarithm of the dividend, minus the logarithm of the divisor.*

Divide (1) (I) by (2), member by member.

III. *The logarithm of any power of a quantity is equal to the logarithm of the quantity itself, multiplied by the exponent of the power.*

IV. *The logarithm of any root of a quantity equals the logarithm of the quantity itself, divided by the index of the root.*

513. Discussion of the equation $a^x = N$.

1°. We will assume that N is always positive.

2°. Suppose $N=1$; then $x=0$, since a^0 is always 1; *i. e.*, The logarithm of unity, in every system, is zero.

3°. Suppose $N=a$; then, since $a^1=a$, $x=1$. Hence,

The logarithm of any base, taken in the system of which it is the base, is unity.

4°. Suppose $a>1$. Then, whatever the power to which it be raised, whether the exponent thereof be entire or fractional, the result will be greater than unity, if this exponent be positive. If this exponent be negative, however, it indicates that the reciprocal of a is to be raised to a power indicated by the positive numerical value of the exponent. Now, as a is greater than 1, $\frac{1}{a}<1$, i. e., is a proper fraction, any power and any root of which is a proper fraction if the exponent of the power, or index of the root, is positive. If, then, $a,>1$, be raised to a power indicated by a negative exponent, the result will be less than 1. We see, then, that if $a>1$, when $N>1$, $x>0$, or positive; when $N=1$, $x=0$; when $N<1$, $x<0$, or negative. Hence,

In a system of which the base is greater than 1, the logarithms of all numbers greater than 1 are positive, and the logarithms of proper fractions are negative.

Suppose that, N being less than 1, x is numerically equal to x' ; whence $a^{-x}=N$. Then, $\frac{1}{a^x}=N$. Suppose N to diminish continually. As N diminishes, its equal, $\frac{1}{a^x}$, must also diminish; and since 1 is invariable, the denominator is the only part of the fraction that can change, and it must therefore increase. In any system, however, the base is fixed; hence, as a cannot change, and is greater than 1, the only way of increasing a^x is to increase x . Hence, as $\frac{1}{a^x}$, or N , diminishes, x must increase; when N is very small x is very great; finally, when N becomes less than any assignable quantity (i. e., 0), x becomes greater than any assignable quantity (i. e., $x=\infty$). Hence, as $x=-x'$, when $N=0$, $x=-\infty$; that is,

In any system of which the base is greater than 1, the logarithm of zero is $-\infty$.

If, a and N being greater than 1, N increases beyond any assignable limit, x will increase beyond any assignable limit; that is, if $N = \infty$, $x = \infty$. Hence,

In any system of which the base is greater than 1, the logarithm of infinity is $+\infty$.

5°. Suppose $a < 1$. a being a proper fraction, if it be raised to any power the exponent of which is positive, the result will be a proper fraction, smaller as the exponent is greater. If the exponent be negative, the reciprocal of the proper fraction, which reciprocal must be an improper fraction, is to be raised to some power with a positive exponent, and the result is therefore greater than 1. If, then, $a < 1$, and $N = 1$, $x = 0$; as N diminishes x increases, and while $N < 1$, x is always greater than 0, i. e., is positive. When N becomes very small, x becomes very large; and finally, when N becomes less than any assignable quantity, x becomes greater than any assignable quantity, i. e., when $N = 0$, $x = +\infty$. If, on the other hand, N becomes greater than 1, x becomes less than zero, i. e., becomes negative, and the farther N recedes from 1, the farther x recedes from 0; i. e., as N increases, x decreases algebraically, and increases numerically. When N becomes very large, x becomes very large numerically, but is negative; finally, when N becomes greater than any assignable quantity, x becomes numerically greater than any assignable quantity, but remains negative; i. e., when $N = \infty$, $x = -\infty$. Hence we conclude that

In any system, of which the base is less than 1, the logarithms of all numbers greater than 1 are negative, and the logarithms of all proper fractions are positive.

In any system, the base of which is less than 1,

The logarithm of zero is $+\infty$, and

The logarithm of infinity is $-\infty$.

THE LOGARITHMIC SERIES.

514. The Logarithmic Series is a series into which the logarithm of a number is developed, so that the terms are arranged according to the ascending powers of a number less by 1 than the original number.

514*. *The logarithm of a number cannot be developed into a series arranged according to the ascending powers of the number.*

For, assume $a^x = y$, whence, $\log_a y = x$. Suppose $\log_a y = P + Qy + Ry^2 + Sy^3 + \dots$. This equation being identical, is true for all values of y ; hence it must be true when $y=0$, i. e., $\log_a 0 = P$, or $\pm\infty = P$ (as $a < 1$ or > 1); a manifest absurdity.

Again, suppose that the first term of the development contains y to any power whatever, i. e.,

$$\log_a y = Qy^m + Ry^{m+1} + Sy^{m+2} + \dots;$$

when $y=0$, this becomes $\log_a 0 = 0$, or $\pm\infty = 0$.

Hence $\log_a y$ cannot be developed into a series arranged according to the ascending powers of y . The development

$$\log_a (1+y) = P + Qy + Ry^2 + \dots$$

presents no difficulty, for when $y=0$, we have $\log_a 1 = P$, i. e., $P=0$, showing that the development must commence with a term containing y , i. e., must be of the form

$$\log_a (1+y) = Qy + Ry^2 + Sy^3 + \dots;$$

in which, if $y=0$, there results $\log_a 1 = 0$ or $0=0$.

515. To Deduce the Logarithmic Series.

Take the exponential equation $a^x = z$, in which a represents any base whatever, and z any proposed quantity. Suppose $a=1+b$ and $z=1+y$; whence $(1+b)^x = 1+y$ (1), or $x = \log_{(1+b)}(1+y)$.

Raising both members of (1) to the power t , we obtain $(1+b)^{xt} = (1+y)^t$ (2); or, expanding each member by the binomial formula, $1 + xtb + \frac{xt(xt-1)}{1.2}b^2 + \frac{xt(xt-1)(xt-2)}{1.2.3}b^3 + \dots$
 $= 1 + ty + \frac{t(t-1)}{1.2}y^2 + \frac{t(t-1)(t-2)}{1.2.3}y^3 + \dots$ (3). **Dividing both**

members by t , after striking out 1 from each, we obtain,

$$xb + \frac{x(xt-1)}{2}b^2 + \frac{x(xt-1)(xt-2)}{1.2.3}b^3 + \dots$$

$$= y + \frac{t-1}{2}y^2 + \frac{(t-1)(t-2)}{1.2.3}y^3 + \dots \quad (4).$$

$$\text{Or, } x\left(b + \frac{xt-1}{2}b^2 + \frac{(xt-1)(xt-2)}{1.2.3}b^3 + \dots\right)$$

$$= y + \frac{t-1}{2}y^2 + \frac{(t-1)(t-2)}{1.2.3}y^3 + \dots \quad (5).$$

This equation being identical, is true for all values of t , and is, therefore, true when $t=0$; then,

$$x\left(b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \dots\right) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \quad (6).$$

$$\text{Or, } x = \log(1+y) = \frac{y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots}{b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \dots} \quad (7).$$

$$\text{Placing } M = \frac{1}{b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \dots} \quad (8), \text{ we obtain,}$$

$$x = \log(1+y) = M\left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \mp \frac{y^n}{n}\right) \quad (9).$$

The Logarithmic Series.

The logarithm of a quantity consists, then, of two factors, one, the factor in parenthesis, depending on the quantity itself, and the other depending on the base of the system, and, therefore, constant for every system. This constant factor, M , is called the **Modulus** of the system.

By giving to the modulus any value, we may deduce the corresponding base. Napier's modulus=1, and the number b , corresponding, is 1.71828; whence a , or the base of his system, is 2.71828.

516. Theorem.—*The logarithms of the same number in any two systems are to each other as the moduli of the systems.*

Let $1+y$ represent the given number, and M and M' the moduli of the systems, a and a' being the bases. Then

$$\log_a(1+y) = M \left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \mp \frac{y^n}{n} \right) \quad (1).$$

$$\log_a(1+y) = M' \left(y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \dots \mp \frac{y^n}{n} \right) \quad (2).$$

The factors in the parentheses being the same, we obtain, dividing (1) by (2), member by member,

$$\frac{\log_a(1+y)}{\log_a(1+y)} = \frac{M}{M'} (3); \text{ or } \log_a(1+y) : \log_a(1+y) :: M : M' \quad (4).$$

517. Corollary I.—*Given the logarithm of a number in any system, the logarithm of this number in any other system may be obtained by multiplying the given logarithm by the modulus of the new system, and dividing the result by the modulus of the original system.*

$$\text{From (3), Art. 516, } \log_a(1+y) = \frac{\log_a(1+y)M'}{M} \quad (1).$$

518. Corollary II.—*The Napierian logarithm of any number, multiplied by the modulus of any other system, will be the logarithm of the given number in this system.*

$$\text{In (1), (518), if } M'=1, \text{ we have, } l(1+y) = \frac{\log_a(1+y)}{M}, \text{ or}$$

$$\log_a(1+y) = M.l(1+y).$$

From this proposition it follows that the modulus of a system may be defined thus:

519. The Modulus of a system is the number by which the Napierian logarithm of a number must be multiplied, to produce the logarithm of the same number in the system proposed; or

The Modulus of any system is the reciprocal of the Napierian logarithm of the base of the system.

Thus, making $y=a-1$ or $y+1=a$, $la = \frac{\log_a a}{M}$; or, since

$$\log_a a = 1, la = \frac{1}{M} \therefore M = \frac{1}{la}.$$

520. Corollary III.—**CONVERSELY.**—*The logarithm of a number in any system, divided by the modulus of this system, is equal to the Napierian logarithm of the number.*

521. Corollary IV.—*The modulus of any system is the logarithm, in that system, of the Napierian base.*

If the base of the Napierian system be denoted by e , we have (516) $\log_a e :: M : 1$ $\therefore$ since $le=1$, (512), $\log_a e=M$.

CALCULATION OF LOGARITHMS.

522. It might be supposed that, by giving a value to M , and substituting for y , in the logarithmic series, values from 0 upward, we could, by summing the second member, find the logarithms of numbers. While this method of proceeding is correct, it is practically useless, as the series does not converge if $y>1$, though if $y<1$ it converges, very slowly. As M in the Napierian system is 1, the simplest method of calculating is to obtain the Napierian logarithms first, after which, logarithms in any other system may be computed in accordance with the principles just discussed. Now, if $M=1$, we have, from (515), (9), for $y=1$,

$$12=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots\mp\frac{1}{n},$$

which converges very slowly. If $y=2$, we have

$$13=2-\frac{2^2}{2}+\frac{2^3}{3}-\frac{2^4}{4}+\frac{2^5}{5}-\dots\mp\frac{2^n}{n},$$

a series in which each term is greater than the preceding, and for greater values of y the divergence is still more marked.

523. To render the logarithmic series sufficiently converging to be of use in calculating logarithms.

We have $\log_a(1+y)=M\left(y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\dots\right)$ (1).

Substitute in this expression $-y$ for y ; there results,

$\log_a(1-y)=M\left(-y-\frac{y^2}{2}-\frac{y^3}{3}-\frac{y^4}{4}-\dots\right)$ (2); whence,

$$\log_a(1+y)-\log_a(1-y)=2M\left(y+\frac{y^3}{3}+\frac{y^5}{5}+\dots\right)$$
 (3).

In this, as $1-y$ is negative for all values of y greater than 1, and negative numbers have no logarithms, we must assign to y such a value that we shall not have the logarithm of a nega-

tive quantity, *i. e.*, we must assign a fractional value. To do this, put the first member under the form, $\log_a \left(\frac{1+y}{1-y} \right)$; now suppose $\frac{1+y}{1-y} = \frac{1+z}{z}$; whence $y = \frac{1}{2z+1}$, a proper fraction for all positive values of z . Substitute in (3), for $\log_a(1+y) - \log_a(1-y)$, its equal, $\log_a(1+z) - \log_a z$, and for y the corresponding value, $\frac{1}{2z+1}$; there results,

$$\log_a(1+z) - \log_a z = 2M \left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \dots \right) \quad (4)$$

or,

$$\log_a(1+z) = \log_a z + 2M \left(\frac{1}{2z+1} + \frac{1}{3(2z+1)^3} + \frac{1}{5(2z+1)^5} + \dots \right) \quad (5)$$

in which the series in the parenthesis converges rapidly, and the more rapidly the greater the value of z . The series has one peculiarity; it fails when $z=0$, so that we cannot obtain the $\log_a 1$ from it. This is no great drawback, however, for we know that the logarithm of 1, in any system, is zero, and (5) does not fail for any other value of z . We might anticipate the failure when $z=0$, for $\log_a 1$ cannot be developed into a series, according to the ascending powers of 1.

524. It will be seen that the transformation of the logarithmic series, so as to render it converging, is merely a clever device for placing y (9) (515) in the form of a fraction, so that the higher the power of y involved in a term, the smaller, numerically, that term may be. A transformation might have been effected by placing, at once, $y = \frac{1}{z}$ in (1), giving $\log_a \left(\frac{1+z}{z} \right) = M \left(\frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \text{etc.} \right)$, or $\log_a(1+z) = \log_a z + M \left(\frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \text{etc.} \right)$, in which we have a series converging (500), but not so rapidly converging as (5). By giving to y various fractional values, various

series could be deduced, any one of which might be used for calculating logarithms, but the series in (5) is the only one ordinarily used.

THE NAPIERIAN SYSTEM.

525. To Calculate Napierian logarithms by this series, (5) (523), make $M=1$, and there results

$$l(z+1)=lz+2\left(\frac{1}{2z+1}+\frac{1}{3(2z+1)^3}+\frac{1}{5(2z+1)^5}+\dots\right)$$

Substituting for z successively the values 1, 2, 4, 6, 8, 10, etc., find the logarithms of the prime numbers; the logarithm of any composite number is (512, I) the sum of the logarithms of its factors. A few Napierian logarithms are here given:

$$11=0 \quad \quad \quad =0.000000.$$

$$12=2\left(\frac{1}{3}+\frac{1}{3 \cdot 3^3}+\frac{1}{5 \cdot 3^5}+\frac{1}{7 \cdot 3^7}+\dots\right) \quad \quad \quad =0.693147.$$

$$13=0.693147+2\left(\frac{1}{5}+\frac{1}{3 \cdot 5^3}+\frac{1}{5 \cdot 5^5}+\frac{1}{7 \cdot 5^7}+\dots\right)=1.098612.$$

$$14=2.12 \quad \quad \quad =1.386294.$$

$$15=1.386294+2\left(\frac{1}{9}+\frac{1}{3 \cdot 9^3}+\frac{1}{5 \cdot 9^5}+\frac{1}{7 \cdot 9^7}+\dots\right)=1.609437.$$

$$16=13+12 \quad \quad \quad =1.791759.$$

$$17=1.791759+2\left(\frac{1}{13}+\frac{1}{3 \cdot 13^3}+\frac{1}{5 \cdot 13^5}+\frac{1}{7 \cdot 13^7}+\dots\right)=1.945910.$$

$$18=14+12=3.12 \quad \quad \quad =2.079441.$$

$$19=2.13 \quad \quad \quad =2.197224.$$

$$110=15+12 \quad \quad \quad =2.302585.$$

Etc., etc., etc.

THE COMMON SYSTEM.

526. While the Napierian system is, on account of its simple modulus, generally used in higher mathematics, the Briggsian, or common system, is peculiarly adapted to ordinary numerical computation, for reasons that will appear as we proceed in its discussion.

527. In the equation, $a^x=N$, the value of x is integral when N is a perfect power of a , and then only. Hence, the logarithms of most numbers must be fractional or mixed, *i. e.*, must consist of an integer (0, 1, 2, 3, etc.), plus a fraction, usually expressed decimally. The integer is called the **Characteristic**, or **Index**, of the logarithm; the decimal part is called the **Mantissa** (from a Tuscan word, signifying *something added on*). One great advantage of the common system is, that in it the characteristic of the logarithm of a number may be written at once when the number is known.

528. To Deduce a Rule for finding the Characteristic of the logarithm of any number.

The base of the system being 10, we have,

$(10)^0=$	1.....	or log	1=0.
$(10)^1=$	10.....	or log	10=1.
$(10)^2=$	100.....	or log	100=2.
$(10)^3=$	1000.....	or log	1000=3.
		log	$10^n=n$.

An examination of this table discloses the fact that the logarithm of any number between 1 and 10 is between 0 and 1, *i. e.*, consists of 0 plus a decimal; the logarithm of any number between 10 and 100 is between 1 and 2, *i. e.*, consists of 1 plus a decimal; the logarithm of any number between 100 and 1000 is between 2 and 3, *i. e.*, consists of 2 plus a decimal; and, in general, the logarithm of any number between 10^n and 10^{n+1} , is between n and $n+1$, *i. e.*, consists of n plus a decimal. Now, any number between 1 and 10 is expressed by one figure before the decimal point; any number between 10 and 100 by two figures; any number between 100 and 1000 by three figures; and, in general, any number between 10^n and 10^{n+1} , by $n+1$ figures. Hence, the characteristic of the logarithm of a number expressed by 1 figure before the decimal point is 0; of a number expressed by two figures before the decimal point, 1;

of a number expressed by three figures, 2; and, in general, of a number expressed by n figures, is $n-1$. Hence the

RULE.—*The characteristic of the logarithm of an integer is positive, and numerically less by 1 than the number of orders of units in the given number.*

A little reflection will show that the number of figures following the decimal point has no effect on the characteristic. Thus, 54.6, 54.62, 54.625, 54.62575, 54.6257392, and 54.62573926482, all lie between 10 and 100, and hence the logarithm of any one lies between 1 and 2, i. e., consists of 1 plus a decimal, so that the characteristic is, in each case 1. As we might reason similarly, on any mixed numbers, we deduce the following

RULE.—*The characteristic of the logarithm of a mixed number is the same as the characteristic of the integral part; that is, it is less by 1 than the number of figures preceding the decimal point.*

To discuss the logarithms of decimals, take the following table:

1.	$= 10^0 = 10^0$	hence, log 1.	$= 0$
.1	$= \frac{1}{10} = 10^{-1}$	log .1	$= -1$
.01	$= \frac{1}{100} = 10^{-2}$	log .01	$= -2$
.001	$= \frac{1}{1000} = 10^{-3}$	log .001	$= -3$
.0001	$= \frac{1}{10000} = 10^{-4}$	log .0001	$= -4$

The logarithm of every number between 1 and .1, lies between 0 and -1 , i. e., is equal to -1 plus a decimal; the logarithm of every number between .1 and .01 lies between -1 and -2 , i. e., equals -2 plus a decimal; the logarithm of any number between .01 and .001 lies between -2 and -3 , i. e., equals -3 plus a decimal, and so on. Hence, as every number between 0 and .1 will have no zero

immediately following the decimal point, every number between .1 and .01 will have *one* zero immediately following the point, and so on, we have the

RULE.—*The characteristic of the logarithm of a decimal fraction is negative, and numerically greater by 1 than the number of zeroes immediately following the decimal point.*

It should be observed that the characteristic *alone* is negative; the mantissa is always positive. This fact is indicated by writing the negative sign *over* the characteristic, thus:

$$\log .0015 = \bar{3}.176091 = -3 + .176091$$

$$\log .00026 = \bar{4}.414973 = -4 + .414973$$

The three foregoing rules may all be expressed in the following

GENERAL RULE.—*The characteristic of the logarithm of a number is always equal to the number of places by which the first significant figure at the left of the number is removed from the units' place, and is positive when this figure is to the left of units, negative when it is to the right, and zero when it is in the units' place.*

529. *The mantissa of the logarithm of a number is not affected by changing the position of the decimal point in the given number; i, e., one mantissa is common to the logarithms of all numbers having the same significant figures.*

Since $\log 10 = 1$, if a number be multiplied or divided by 10, its logarithm will be increased or diminished by 1, which addition or subtraction can affect only the characteristic. If the number be multiplied or divided by 100, the characteristic of the logarithm will be increased or diminished by 2, but the mantissa will remain unchanged. In general, if a number be multiplied or divided by any power of 10, the logarithm of the number will be increased or diminished by the exponent of the power, and as the power involved in a change of the decimal point has always an integral exponent, this addition or subtraction can affect only the characteristic.

If $\frac{c}{10^q}$ represent a number having q decimal places, $\frac{c}{10^{q \pm p}}$ will represent the number when the decimal point is removed p places to the left or right. Now $\log \frac{c}{10^q} = \log c - q$,

and $\log \frac{c}{10^{q \pm p}} = \log c - (q \pm p)$, and as q and p are both integral, the mantissa in each case will be the same as the mantissa of the logarithm of c . We see, then, that

The mantissa of the logarithm of a decimal fraction is the same as the mantissa of the logarithm of the numerator.

To find the logarithm of a common fraction we may subtract the logarithm of the denominator from the logarithm of the numerator, or we may reduce the fraction to a decimal and then take the logarithm as here indicated.

530. The common system is the only one in which the characteristics of logarithms may be found without reference to tables, and the only system in which logarithms of fractions may be obtained directly from the tables.

TABLES OF LOGARITHMS.

531. A Table of Logarithms is a table containing a list of numbers and their logarithms, so arranged that we may find therefrom the logarithm of any given number, and the number corresponding to any given logarithm.

The tables usually contain the numbers from 1 to 10000, with their logarithms and **Tabular Differences**, enabling us to obtain from them the logarithm of any number. In common tables mantissas only are given, the characteristics being found by the rules already laid down. For further information concerning the tables, see page i.

532. To Compute a Table of Common Logarithms. Tables of common logarithms may be computed in several ways, some of which will be noted hereafter. The most expeditious is, perhaps, the method by the use of (5) (523),

in which, for M , the modulus of the common system is substituted. Or, having a table of Napierian logarithms, computed as in (525) or otherwise, the corresponding common logarithms may be obtained by multiplying the logarithms in this table by the modulus of the common system. In either case the modulus of the common system must be known.

533. To find the Modulus of the Common System (519) is employed.

The modulus of any system being the reciprocal of the Napierian logarithm of the base of the system, we have

$$M = \frac{1}{\ln 10}.$$

$$\text{Now } \ln 10 = 2.302585093 \therefore M = \frac{1}{2.302585093} = 0.434294482.$$

534. To find the Base of the Napierian System.—If e be this base, we have (521) $\log e = 0.434294482$, whence, from a table of common logarithms, finding the number corresponding to this logarithm, we have $e = 2.718281828 \dots$, or $e = 2.71828$. This is approximately the sum of the series

$$1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \dots$$

APPLICATIONS

OF THE THEORY OF LOGARITHMS.

535. The manner in which numerical calculations are simplified by the use of logarithms has been already indicated (512). When several divisors and multipliers are to be used, the principle of the *Arithmetical Complement* affords a further means of simplifying the operation.

The **Arithmetical Complement** of a number, with respect to another, is the difference between that other and the given number: thus, a. c. 5, with respect to 9, is 4; a. c. 6, with respect to 10, is 4, etc. The arithmetical comple-

ments of logarithms are always taken with respect to 10, and their use in numerical calculation is based on the following

Proposition.—*The logarithm of any quotient equals the sum of the logarithm of the dividend and the arithmetical complement of the logarithm of the divisor, diminished by 10.*

$\log \frac{a}{b} = \log a - \log b$ (512). Since 10 may be added to, and subtracted from, the second member, without altering its value, $\log a - \log b = \log a + (10 - \log b) - 10$; or, since $10 - \log b = \text{a. c. log } b$, $\log a - \log b = \log a + \text{a. c. log } b - 10$. Hence, $\log \frac{a}{b} = \log a + \text{a. c. log } b - 10$.

TO ILLUSTRATE this by an example: Solve the proportion,

$$89721:358884::5672:x; \text{ we have } x = \frac{358884 \times 5672}{89721}$$

$$\log 358884 = 5.554954$$

$$\log 5672 = 3.753736$$

$$\text{a. c. log } 89721 = 5.047106$$

$$\log x = 4.355796 \quad \therefore x = 22688.$$

By the use of this principle, we are enabled to dispense with subtraction altogether, which, in a problem requiring many divisions, is a great convenience.

Logarithms are very useful in operations requiring the extraction of roots with large indices, and in raising to powers with large integral or with fractional exponents, and examples illustrating their applicability to problems in percentage, interest, etc., will be given hereafter.

Before solving the following illustrative examples, the student should learn thoroughly the manner of using the Table of Logarithms: see page i.

EXAMPLES.

$$1. \sqrt[3]{\frac{-(37.21)^{1.3} + 4327.03 \sqrt[3]{(0.0672304)^{-0.7}}}{\sqrt[3]{0.0000273} + 10 \sqrt[3]{(0.3986)^{3.3} \sqrt[3]{(0.0087)^{-3.5}}}}} = ?$$

Denote by $-x$ the first term, and by y the second term, of the numerator of the fraction under the sign, and by z

and y' the corresponding terms of the denominator. Then
 $\log x = \log 37.21 \times 1.3 = 1.57066 \times 1.3 = 2.041858$.

$$\begin{array}{r} 2.041858 \quad N=1101 \\ 2.041787 \quad \text{Diff.} = 393 \therefore x = 110.1181: \\ \hline 393 \mid 71 \mid 181 \end{array}$$

$$\log (0.0672304)^{-0.7} = 2.827566 \times -0.7 = -1.172434 \times -0.7 \\ = 0.8207038.$$

$$\begin{array}{r} 0.8207038 \div 3.2 = 0.2564699 \\ \log 4327.03 = 3.6361900 \\ \hline \log y = 3.8926599 \quad N=7810, \\ 3.8926510 \quad \text{Diff.} = 56. \\ \hline 56 \mid 89 \mid 15893 \therefore y = 7810.15893. \end{array}$$

$$\begin{array}{r} \text{The numerator, } y-x, = 7810.15893 - 110.1181, \\ = 7700.04083. \end{array}$$

In the denominator:

$$\begin{array}{r} \log x' = \log 0.0000273 \div -\frac{3}{2} = 5.436163 \div -\frac{3}{2} \\ = -4.563837 \div -\frac{3}{2} = 3.042558 \quad N=1102 \\ 3.042182 \quad \text{Diff.} = 393 \\ \hline 393 \mid 376 \mid 9567 \\ \therefore x' = 1102.9567 \end{array}$$

$$\begin{array}{r} \log \sqrt[0.05]{(0.3986)^{3.2}} = \frac{\log 0.3986 \times 3.2}{0.05} = \frac{.1.600537 \times 3.2}{0.05} \\ = \frac{-0.399463 \times 3.2}{0.05} = \frac{-1.2782816}{0.05} = -25.565632 \end{array}$$

$$\begin{array}{r} \log \sqrt[0.3]{(0.0087)^{-3.5}} = \frac{\log 0.0087 \times -3.5}{0.3} = \frac{3.939519 \times -3.5}{0.3} \\ = \frac{-2.060481 \times -3.5}{0.3} = \frac{7.2116835}{0.3} = \frac{24.038945}{-25.565632} \\ \log y' = -1.526687 \\ = 2.473313 \end{array}$$

$$2.473313$$

$$N=2973$$

$$2.473195$$

$$\text{Diff.} = 146$$

$$\hline 146 \mid 118 \mid 8082$$

$$\therefore y' = 0.02974 \times 10 = 0.2974$$

Hence the denominator is $1102.9567 + 0.2974$; or, 1103.2541.

$$\log 7700.04083 = 3.886493287$$

$$\log 1103.2541 = 3.042675861$$

$$\text{logarithm of fraction} = 0.843817426$$

$$0.843817426 \div -302 = -.0027940974 = \bar{1}.9972059026.$$

The number corresponding to this logarithm is 0.99358614, which is therefore the required root.

$$2. \sqrt[10]{\frac{(2.536)^2 + 489.62(3.786)^{-2.5}}{.000572 \sqrt[2.1]{.00013579}}} = ?$$

Denote the second term of the numerator of the fraction under the sign by x . $\log x = \log 489.62 + \log (3.786)^{-2.5} = 2.689859 + 0.578181 \times -2.5 = 2.689859 - 1.4454525 = 1.2444065 \therefore x = 17.55522.$

$$\log (2.536)^2 = .404149 \times 2 = .808298 \therefore (2.536)^2 = 6.43128$$

$$17.55522$$

$$\text{Numerator of fraction} = 23.98650$$

$$\text{Further, } \log (.000572 \sqrt[2.1]{.00013579}) =$$

$$\log .000572 + \frac{\log .00013579}{2.1} = \bar{4}.757396 + \frac{\bar{4}.132869}{2.1}$$

$$= -3.242604 - \frac{3.867131}{2.1} = -5.084095$$

$$\text{Logarithm of Numerator of fraction} = 1.379967$$

$$\text{Logarithm of Denominator of fraction} = -5.084095$$

$$\text{Logarithm of fraction} = 6.464062$$

Dividing this logarithm by 10, we obtain 0.6464062 as the logarithm of the number required. Hence, 4.4300224, the number corresponding to this logarithm, is the required root.

$$3. \sqrt[15]{\frac{0.0062.(3.557)^{-\frac{2}{3}}}{126^{\frac{3}{2}} \sqrt[0.00000137]{}}} = ?$$

$$4. \sqrt[5]{\frac{7.284 + 497.3 \times (4.289497)^{-2.1}}{0.0479 \sqrt[0.001593478]{}^2}} = ? \quad \text{Ans. } 8.594815.$$

$$5. \sqrt[5]{\sqrt{1+\sqrt{0.2}}-\sqrt{0.5+\sqrt{0.2}}}=? \quad \sqrt[9]{\frac{0.12 \sqrt{0.4}}{0.4+\sqrt{0.4}}}=?$$

$$\sqrt[3]{\frac{3(2.4)^{3.1}}{\sqrt{0.24}}}=? \quad \text{Ans. } 0.7451646; 0.7482345; 4.520885.$$

$$6. \sqrt[301]{0.007251}=? \quad \sqrt[20]{\frac{5.72(3.567)^{-0.01}}{\sqrt[100]{5.82} \sqrt[3]{5763}}}=?$$

EXPONENTIAL EQUATIONS.

536. The use of logarithms greatly facilitates the solution of exponential equations. Take the equation $a^x=b$. If x is entire, its value may be found by trial; thus, if $3^x=243$, we see that $3^1=9$, $3^4=81$, $3^5=243$, whence $x=5$.

537. If x is not entire, the equation may be solved in two general ways: by the use of continued fractions, and by the means of logarithms. The former method will be illustrated first.

Take the equation $2^x=10$ (1); x lies between 3 and 4, for $2^3=8$, and $2^4=16$. Place, then, $x=3+\frac{1}{x'}$. Substituting in (1), $2^{3+\frac{1}{x'}}=10$, or $2^3 \times 2^{\frac{1}{x'}}=10$, i. e., $8 \times 2^{\frac{1}{x'}}=10$. $\therefore 2^{\frac{1}{x'}}=\frac{10}{8}=\frac{5}{4}$ (2). Raising the members of (2) to the power x' , we obtain $\left(\frac{5}{4}\right)^{x'}=2$ (3), an equation which, being similar to (1),

may be similarly treated. As $\left(\frac{5}{4}\right)^3=\frac{125}{64}<2$, and $\left(\frac{5}{4}\right)^4=\frac{625}{256}>2$, x' lies between 3 and 4. Let $x'=3+\frac{1}{x''}$. Then $\left(\frac{5}{4}\right)^3 \times \left(\frac{5}{4}\right)^{\frac{1}{x''}}=2$; $\frac{125}{64} \cdot \left(\frac{5}{4}\right)^{\frac{1}{x''}}=2$; $\left(\frac{5}{4}\right)^{\frac{1}{x''}}=\frac{128}{125}$ (4).

Raising both members of (4) to the power x'' , we obtain, $\left(\frac{128}{125}\right)^{x''}=\frac{5}{4}$ (5), which, being of the same form as (1) and (3), may be treated as they have been. Here x'' lies be-

tween 9 and 10; let $x' = 9 + \frac{1}{x''}$. Then, $\left(\frac{128}{125}\right)^x \times \left(\frac{128}{125}\right)^{\frac{1}{x''}}$
 $= \frac{5}{4}$, or $\left(\frac{128}{125}\right)^{\frac{1}{x''}} = \frac{5(125)^x}{4(128)^x}$ (6), from which the limits of x''
 may be determined. In a similar way the process may be
 continued as far as is desirable, or as far as necessary to
 secure any required degree of accuracy. Replacing x' and
 x'' by their values, we find $x = 3 + \frac{1}{3 + \frac{1}{9 + \frac{1}{x'''}}}$

or, $x = 3 + \frac{9}{28} = \frac{93}{28} = 3.321+$, to within .001, or, $2^{x.321} = 10$.

Take, as a further example, the equation $3^x = 6$; x being
 > 1 and < 2 , place $x = 1 + \frac{1}{x'}$. Then, $3 \cdot 3^{\frac{1}{x'}} = 6 \therefore 3^{\frac{1}{x'}} = \frac{6}{3} = 2$,
 and raising both members to the power x' , $2^{x'} = 3$; x' being
 between 1 and 2, place $x' = 1 + \frac{1}{x''}$; whence,

$$2 \cdot 2^{\frac{1}{x''}} = 3 \therefore 2^{\frac{1}{x''}} = \frac{3}{2}, \text{ and } \left(\frac{3}{2}\right)^{x''} = 2. \text{ Now, } \left(\frac{3}{2}\right)^1 < 2,$$

and $\left(\frac{3}{2}\right)^2 = \frac{9}{4} > 2$. x'' being between 1 and 2, place

$$x'' = 1 + \frac{1}{x'''}; \text{ whence, } \frac{3}{2} \cdot \left(\frac{3}{2}\right)^{\frac{1}{x'''}} = 2; \therefore \left(\frac{3}{2}\right)^{\frac{1}{x'''}} = \frac{4}{3}, \text{ and}$$

$$\left(\frac{4}{3}\right)^{x'''} = \frac{3}{2}; \left(\frac{4}{3}\right)^1 < \frac{3}{2}, \left(\frac{4}{3}\right)^2 = \frac{16}{9} > \frac{3}{2}. \text{ Place, then, } x''' = 1 + \frac{1}{x^{iv}};$$

$$\text{whence, } \frac{4}{3} \cdot \left(\frac{4}{3}\right)^{\frac{1}{x^{iv}}} = \frac{3}{2} \text{ and } \left(\frac{4}{3}\right)^{\frac{1}{x^{iv}}} = \frac{9}{8}.$$

Raising both members to the x^{iv} th power,

$$\left(\frac{9}{8}\right)^{x^{iv}} = \frac{4}{3}; \text{ as } \left(\frac{9}{8}\right)^2 = \frac{81}{64} < \frac{4}{3}, \text{ and } \left(\frac{9}{8}\right)^3 = \frac{729}{512} > \frac{4}{3},$$

x^{iv} is between 2 and 3; place, then, $x^{iv} = 2 + \frac{1}{x^v}$, and there

$$\text{results } \left(\frac{9}{8}\right)^2 \cdot \left(\frac{9}{8}\right)^{\frac{1}{x^v}} = \frac{4}{3}, \text{ or } \frac{81}{64} \cdot \left(\frac{9}{8}\right)^{\frac{1}{x^v}} = \frac{4}{3}, \text{ and } \left(\frac{9}{8}\right)^{\frac{1}{x^v}} = \frac{256}{243},$$

$$\text{or } \left(\frac{256}{243}\right)^{x^v} = \frac{9}{8}, \text{ in which } x^v = 2 + \frac{1}{x^v}, \text{ and therefore,}$$

$$\left(\frac{256}{243}\right)^2 \cdot \left(\frac{256}{243}\right)^{\frac{1}{x^v}} = \frac{9}{8}, \text{ and by continuing the process, } x^v, x^{vi},$$

etc., may be found. Replacing x' , x'' , etc., by their values we have,

$$x = 1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{x^{11}}}}}}}} = 1.6315 +$$

EXAMPLES.

Solve the following exponential equations:

$$3^x = 20. \quad \text{Ans. } 2.727. \quad 3^x = 2546. \quad \text{Ans. } 7.138.$$

$$10^x = 4. \quad \text{Ans. } 0.602. \quad 2^x = 10.$$

$$5^x = \frac{2}{3}. \quad \text{Ans. } -0.25. \quad 6^x = \frac{3}{4}.$$

In general, to develop any quantity, as x , into a continued fraction, place $x = a + \frac{1}{x'}$; $x' = b + \frac{1}{x''}$, $x'' = c + \frac{1}{x'''}$, etc., a being the greatest integer in x , b the greatest integer in x' , c the greatest integer in x'' , and so on. The numbers, a , b , c , ..., being found, and x' , x'' , x''' , etc., being replaced by their values, there results, $x = a + \frac{1}{b + \frac{1}{c + \frac{1}{d + \dots}}}$.

It will be seen that the operation just outlined is one method of computing logarithms, the number of which x is the exponent being the base of the system.

538. Solution by means of logarithms. Taking the logarithm of each member of the equation $a^x = b$, we obtain $x \log a = \log b$; whence, $x = \frac{\log b}{\log a}$. If x is a polynomial containing but one unknown quantity, an equation of the degree of this polynomial must be solved; if x be a polynomial containing more than one unknown quantity, the equation is indeterminate. Thus, in the equation $5^{x^2-7x+11} = \frac{1}{5}$, we have $x^2 - 7x + 11 = -1$, or $x^2 - 7x = -12$; whence, $x = 3$ or 4 .

Again, take the equation $Pa^x + Qa^{x-1} + Ra^{x-2} + \dots = M$. Factoring, we obtain $a^x \left(P + \frac{Q}{a} + \frac{R}{a^2} + \dots \right) = M$; as a, b, c , etc., and P, Q , etc., are known, denote the factor in parenthesis by F . Then $Fa^x = M$, and $a^x = \frac{M}{F}$, an equation that may be solved in the manner already indicated.

EXAMPLES.

$$1. \left(\frac{.4742759}{4875.3} \right)^x = 0.0459914;$$

$$x = \frac{\log .0459914}{\log .4742759 + a. c. \log 4875.3 - 10}.$$

$$2. 9.(3)^{x+1} = 27^{x-1}; 16(2)^{x+1} = 4^{x+1}. \quad x=3; x=0.$$

$$3. 2^x 3^{x-1} = 100. \quad x=3.18334.$$

$$4. 2^{x-1} + 3^{x+1} = 3^{x-1} + 2^{x+2}; \text{ this may be written}$$

$$\frac{1}{2} 2^x + 3 \cdot 3^x = \frac{1}{3} 3^x + 4 \cdot 2^x \therefore 16 \cdot 3^x = 21 \cdot 2^x \therefore \left(\frac{3}{2} \right)^x = \frac{21}{16}$$

$$\therefore x = 0.67067.$$

$$5. a^x - 2b = ca^x; \text{ place } a^x = y, \text{ whence } a^{-x} = \frac{1}{y}; \text{ then}$$

$$y - 2b = \frac{c}{y}, y^2 - 2by = c, \text{ and } y = b \pm \sqrt{b^2 + c} \therefore a^x = b \pm \sqrt{b^2 + c},$$

$$\text{and } x = \frac{\log (b \pm \sqrt{b^2 + c})}{\log a}.$$

$$6. c^x d^{x-1} e^{x-2} = \frac{abc^x}{d^{x+2}} (1). \text{ Taking the logarithms of the members of (1), } x \log c + (x-1) \log d + (x-2) \log e = \log a + \log b + x \log e - (x+2) \log d; \text{ whence,}$$

$$x = \frac{\log a + \log b - \log d + 2 \log e}{\log c + 2 \log d}.$$

$$7. \sqrt[3]{abc} \div \frac{c^{x+1}}{d} = \frac{dc^{x-1}}{a^2}.$$

$$y = \frac{\log a - \log e}{\log c + \log e} \pm \sqrt{\frac{\log a + \log b + \log c}{\log c + \log e} + \left(\frac{\log a - \log e}{\log c + \log e} \right)^2}.$$

CHAPTER XII.

SECTION I.

ARRANGEMENTS, PERMUTATIONS, AND COMBINATIONS.

539. Permutations are the different orders in which a number of things may be placed; the results obtained by placing a number of things in every possible order. Each thing must, therefore, enter each result, and enter but once. The number of permutations of m things will be indicated thus: P_m .

540. Arrangements are the results obtained by placing a number of things in sets of 1 and 1, 2 and 2, 3 and 3, etc., n and n , the things in each set having every possible order. The number of arrangements of m things taken n in a set will be indicated thus: A_m^n .

541. Combinations are the different groups that can be formed by placing a given number of things in sets of 1, 2, 3, ... n . The groups being *different*, any one must differ from any other by at least one of the things entering them, and the different orders of the members of the groups will not affect the number of combinations. The number of combinations of m things taken n in a set will be indicated thus: C_m^n .

If the things taken be literal factors, it will be seen that combinations correspond to products of the factors taken n at a time, arrangements to the orders of the factors in these products, and permutations to continued products of all the factors taken.

542. To find the number of Arrangements of m things, taken n in a set.

1°. Take three things, as three letters, which, if taken a 1 in a set, give three arrangements. For each arrange- b ment there will be two reserved letters, *i. e.*, letters c that do not enter it. If, now, to a , we annex in turn the reserved letters, b and c , we obtain two sets of let- ab ters having two in each set, ab and ac ; by annexing ac to b , successively, the reserved letters a and c , we ob- ba tain the sets ba and bc , and from c we obtain ca and bc cb , thus obtaining, in all, six arrangements, of three ca letters, taken two in a set. No other arrangements, cb two in a set, can be obtained from these letters, for a occupies all the positions that it can possibly have, with respect to b and c , and, similarly, b and c with respect to a . Hence, as each of the arrangements in sets of 1, gives as many arrangements in sets of two as there are reserved letters, the number of arrangements of three letters, taken two in a set, equals the number of arrangements in sets of 1, multiplied by the number of reserved letters.

If, now, a fourth letter, d , be introduced, and it be required to find the number of arrangements of the four letters, taken three in a set, we may arrange them, first a one in a set. For each set there are three reserved b letters, for a , b , c , and d ; for b , a , c , and d ; for d , a , c b and c ; and as each of the reserved letters can be d annexed to the set from which it is reserved, the number of arrangements of the four letters, in sets of two, is equal to the number of arrangements in sets of one, multiplied by the number of reserved letters, thus:

ab	ba	ca	da
ac	bc	cb	db
ad	bd	cd	dc

Now, for each of these sets of two there are two reserved letters; for ab , c and d ; for ac , b and d , etc. If, therefore, we annex successively to each set the letters reserved from

that set, we shall obtain from each as many arrangements in sets of three, as there are reserved letters, thus:

<i>abc</i>	<i>bac</i>	<i>cab</i>	<i>dab</i>
<i>abd</i>	<i>bad</i>	<i>cad</i>	<i>dac</i>
<i>acb</i>	<i>bca</i>	<i>cba</i>	<i>dbc</i>
<i>acd</i>	<i>bcd</i>	<i>cdb</i>	<i>dba</i>
<i>adb</i>	<i>bda</i>	<i>cda</i>	<i>dca</i>
<i>adc</i>	<i>bdc</i>	<i>cdb</i>	<i>dcb</i>

An examination of this table will show, also, that no more arrangements can be obtained from each set than there are reserved letters, without repetition. Hence, the number of arrangements of four letters, taken three in a set, equals the number of arrangements of four letters, taken two in a set, multiplied by the number of reserved letters.

In general, if m things be given, the number of arrangements in sets of 1 is m , and there will evidently be $m-1$ reserved things. As each reserved thing, in succession, may be placed after the set from which it is reserved, each arrangement in sets of 1 will give $m-1$ arrangements in sets of two. Further, as there are two things in each of these arrangements, there must be $m-2$ reserved things; and, as each reserved thing may be placed after any set of 2 from which it is reserved, each set of 2 will give $m-2$ sets of three, and, therefore, the $m(m-1)$ sets of 2 must give $m(m-1)(m-2)$ sets of three, in all. Similarly, there would be $m(m-1)(m-2)(m-3)$ arrangements in sets of four, and we might conclude at once that the number of arrangements of m things in sets of n , is $m(m-1)(m-2) \dots (m-n+1)$. But to remove all doubt, let us assume that the law already indicated holds good for $\overset{n}{\mathbf{A}}$; then must it hold good for $\overset{n+1}{\mathbf{A}}$. For, from each set of $\overset{n}{\mathbf{A}}$ there will be $m-n$ reserved. Hence, if, after the first set of n , we place, successively, the reserved things, there will result $m-n$ arrangements, in sets of $n+1$, each differing from the others by at least one of the first n things. Similarly, $m-n$ new arrangements can be obtained from the second of the old ones, and so on for each of the arrange-

ments in sets of n . Hence, the number of arrangements of m things in sets of $n+1$, equals the number of arrangements of m things in sets of n , multiplied by the number of reserved letters, that is,

$$\overset{n+1}{\underset{m}{A}} = m(m-1)(m-2) \dots (m-(n-1))(m-n) \quad (1);$$

In other words, if the law indicated holds good for $\overset{n}{\underset{m}{A}}$, it must hold good for $\overset{n+1}{\underset{m}{A}}$. It has been proved to hold good for $\overset{1}{\underset{m}{A}}$, $\overset{2}{\underset{m}{A}}$, $\overset{3}{\underset{m}{A}}$, and therefore it holds good for $\overset{4}{\underset{m}{A}}$, and hence for $\overset{5}{\underset{m}{A}}$, and so on; that is, it is general. Hence,

$$\overset{n}{\underset{m}{A}} = m(m-1)(m-2)(m-3) \dots (m-n+1) \quad (2); \text{ or,}$$

The number of arrangements of m things taken n in a set, is equal to the continued product of the natural numbers from m down to $m-n+1$, inclusive.

543. Corollary.—*The number of Permutations of m things is equal to the continued product of the natural numbers from 1 up to m inclusive.*

If m things be arranged in sets of m , i. e., if each thing enters every arrangement once, the arrangements become permutations, all the things being taken every time. If in (2) (542), $n=m$, $\overset{m}{\underset{m}{A}} = m(m-1)(m-2)(m-3) \dots 2.1$, or, since $\overset{m}{\underset{m}{A}} = \overset{m}{\underset{m}{P}}$, and the order of the factors in the second member may be changed at pleasure,

$$\overset{m}{\underset{m}{P}} = 1.2.3 \dots (m-3)(m-2)(m-1)m \quad (1).$$

544. To find the number of Combinations of m things taken n in a set.

Suppose the number of combinations is C . Now, in each combination there are n things, say letters, and hence each combination may be written in as many different ways, the things in it may be placed in as many different orders, as

there are permutations of n things. From the definition of arrangements, each one of these results would be an arrangement, and therefore one combination will give as many arrangements in sets of n as there are permutations of n things; hence C combinations will give $C(1.2.3\dots n)$ arrangements, and these will be all the arrangements possible of m things, taken in n a set; that is,

$$\overset{n}{A} = \overset{n}{C} (1.2.3\dots n), \text{ or } \overset{n}{C} = \frac{m(m-1)(m-2)\dots(m-n+1)}{1.2.3\dots n}.$$

545.—Corollary. *The number of combinations of m things, taken n in a set, is equal to the number of combination of m things taken $m-n$ in a set.*

$$\overset{n}{C} = \frac{m(m-1)(m-2)\dots(m-n+1)}{1.2.3\dots n} \quad (1).$$

$$\overset{m-n}{C} = \frac{m(m-1)(m-2)\dots(n+1)}{1.2.3\dots(m-n)} \quad (2).$$

Hence,

$$\overset{n}{C} \div \overset{m-n}{C} = \frac{m(m-1)(m-2)\dots(m-n+1)(m-n)\dots 3.2.1}{m(m-1)(m-2)\dots(n+1)n\dots 3.2.1} \quad (3).$$

The numerator and denominator in the second member of (3) being each the product of the natural numbers from m to 1, inclusive, are equal. Hence, $\overset{n}{C} \div \overset{m-n}{C} = 1$; or $\overset{n}{C} = \overset{m-n}{C}$.

546. It will be seen that the expression for $\overset{n}{C}$ is the same as the coefficient of the *general term* of the *binomial formula*, and any coefficient in this formula is, therefore, the number of combinations of m things, taken as many in a set as there are preceding terms. Moreover, since the $(n+1)th$ term from the beginning has n terms before it, and the $(n+1)th$ term from the end has n terms after it, or $m-n$ before it, the corollary furnishes another method of proving that the coefficients of terms equally distant from the extremes are equal. The theory of combinations is the basis of one of the most elegant demonstrations of the binomial formula, which is given in many of our text-books.

The applications of the theory of permutations, combinations, and arrangements, are frequent and important. While one of the most noticeable is in the determination of the coefficients in a development of the form, $A+Bx+Cx^2+Dx^3+\dots$, the applications in geometry are hardly less useful, and the same may be said of the applications in crystallography, in determining the numerous forms of crystals, in chemistry, acoustics, etc. In common life, however, the most important use is in the calculus of probabilities, upon a short discussion of which we are about to enter.

SECTION II.

CALCULUS OF PROBABILITIES.

547. The Mathematical Probability of the occurrence of an event is the ratio of the number of opportunities of occurrence to the whole number of possible cases, favourable and unfavourable.

548. The Mathematical Improbability of the occurrence of an event is the ratio of the number of chances against the occurrence to the whole number of chances for and against the occurrence.

It must be understood, however, that no one case is more likely to present itself than any other, or to prevail over any other. That is, the chances must all be equal in influence.

549. A Simple Probability is the probability of the occurrence of a single event.

550. A Compound, or, as it is often called, **Complex Probability**, is the probability of the simultaneous occurrence of two or more separate and independent events.

551. The Odds in Favour of an occurrence is the ratio of the number of favourable to the number of unfavourable chances; the **Odds Against** an occurrence is the reciprocal of this ratio.

SIMPLE PROBABILITY.

552. If there are h ways in which an event may happen, and f ways in which it may fail, and no one of these cases is any more likely to occur than any other, the probability of the occurrence of the event is $\frac{h}{h+f}$. It is clear that this will be greater, as f decreases, and the denominator thus approaches h in value. If f should be zero, the ratio would be one of equality, would equal 1, and the probability would pass into certainty, as all the h possible cases would be favourable, and therefore the event must occur. 1, then, may be called the *symbol of certainty*. If, on the other hand, f increases, the probability $\frac{h}{h+f}$ diminishes, and when f becomes very large this probability becomes very small. When $f=\infty$, the probability becomes $\frac{h}{\infty}$, i. e., zero.

The probability that the event will not happen, i. e., the probability of failure, is $\frac{f}{h+f}$, which, if h remains unchanged, evidently decreases or increases as f increases or decreases, but not in the same proportion. If (whether f be fixed or not), h increases or decreases, the probability will decrease or increase; when $h=0$, the probability becomes $\frac{f}{f}=1$, i. e., *certainty*.

Again, suppose that there are n events, one of which must occur, and only one of which can occur. Suppose that $a, b, c, \dots$, are the number of ways in which these events can occur, respectively. Then, as the first event can occur in a out of $a+b+c+d+\dots$ ways, the probability of its occurrence is $\frac{a}{a+b+c+d+\dots}$; the probability of the occurrence of the second event is $\frac{b}{a+b+c+\dots}$; and so for every one of the n events. In general, to find the probability of occurrence of any event, we have the

RULE.—*Divide the number of chances for the occurrence by the whole number of chances.*

EXAMPLES.

1. What is the probability that a ball drawn, at random, from a box containing 10 white balls and 25 red ones, shall be white? What is the probability of its being red?

2. If \$600 depends on the turning up of any particular face of a die, what is a single chance worth? *Ans.* \$100.

3. The *Expectation* of a person is the product of the probability of his success in any venture, and the sum to be received in case of success. Suppose that a prize of \$500 is to be disposed of at a lottery. If there are m persons, each of whom has one ticket, what is the expectation of each?

4. A man owning 400 horses and 200 mules is informed by his superintendent that 10 head of stock have been lost. What is the probability of the lost stock being all horses? of its being all mules?

5. What is the probability of throwing at least one ace, with three dice, at a single throw?

As $\frac{h}{h+f} + \frac{f}{h+f} = 1$, the sum of the simple probability and improbability of an event is *certainty*, which would be expected, since a single event must either happen or fail.

COMPOUND PROBABILITY.

553. To find the Compound probability of several independent events.

Suppose, first, that there are but two events, and that the first may occur in h ways, and may fail in f ways; and, further, that there are h' ways in which the second may occur, and f' ways in which it may fail. Now, the first event occurring in any one of the h ways, the second may occur in all the h' ways. Hence, for *any one* favourable

case of the first there are h' , and for *all* the favourable cases of the first hh' , favourable chances of the simultaneous occurrence of the two events. Similarly $(h+f)(h'+f')$ is the whole number of possible cases, favourable and unfavourable. The probability of the simultaneous occurrence of the events is therefore $\frac{hh'}{(h+f)(h'+f')}$. In like manner we

find, for the probability of simultaneous failure, $\frac{ff'}{(h+f)(h'+f')}$.

Suppose that there are three events. Suppose that the first, second, and third may happen in a , b and c ways, respectively, and may fail in a' , b' and c' ways, respectively, all of these cases being equally likely to occur. Then the whole number of chances favourable to simultaneous occurrence is abc , and the number favourable to simultaneous failure, $a'b'c'$. Hence, we may find, as above, for the probability of simultaneous occurrence, $\frac{abc}{(a+a')(b+b')(c+c')}$,

and for the probability of simultaneous failure, $\frac{a'b'c'}{(a+a')(b+b')(c+c')}$,

and similar reasoning may be extended to the case of any number of events. As,

$$\frac{hh'}{(h+f)(h'+f')} = \frac{h}{h+f} \cdot \frac{h'}{h'+f'};$$

$$\frac{abc}{(a+a')(b+b')(c+c')} = \frac{a}{a+a'} \cdot \frac{b}{b+b'} \cdot \frac{c}{c+c'}, \text{ etc.,}$$

the compound probability of several independent events is the product of the simple probabilities of these events. Hence, we deduce, for finding the compound probability of several events, the following

RULE.—Take the continued product of the simple probabilities of the events.

554. While the sum of the simple probability of an event, and the probability of its failure is 1, the symbol of certainty, the sum of the expressions given, (553), as the compound probabilities of simultaneous occurrence and fail-

ure, is not 1. This arises from the fact that, taking the first case, $\frac{ff'}{(h+f)(h'+f')}$ is not the improbability, or *probability of failure*, of the concurrence under discussion, but the probability of *simultaneous* failure of the events. The whole number of chances is $(h+f)(h'+f') = hh' + fh' + f'h + ff'$, and deducting the number of chances (hh') , favourable to the concurrence in question, we obtain $fh' + f'h + ff'$, as the number of chances not favourable to this concurrence. Hence the improbability of the simultaneous occurrence of the events is $\frac{fh' + f'h + ff'}{(f+h)(f'+h')}$, which, added to $\frac{hh'}{(h+f)(h'+f')}$, gives 1, the symbol of certainty, as it should. $\frac{fh'}{(h+f)(h'+f')}$ is the probability that the first event will fail, and the second occur; $\frac{f'h}{(h+f)(h'+f')}$, the probability that the first will occur and the second fail, and $\frac{ff'}{(h+f)(h'+f')}$, as before shown, the probability that the events will fail together. The case of three or more events may be treated in a similar manner.

555. Suppose that there are two dependent events, of which the first may happen in h ways, and may fail in f ways, and of which the second may, after the occurrence of the first, happen in h' ways, and fail in f' ways. Then we may show, just as in (553), that the probability of concurrence of these events is $\frac{hh'}{(h+f)(h'+f')} = \frac{h}{h+f} \cdot \frac{h'}{h'+f'}$; i. e., the product of the probability of the occurrence of the first into the probability that, after the first has occurred, the second will follow. A similar course of reasoning might evidently be applied to the case of three or more dependent events.

EXAMPLES.

1. Required, the probability of throwing 5 aces with 5 dice, at a single throw; also, the probability that not one ace will be thrown. What is the odds against each event?

2. From a box containing 30 tags, numbered from 1 to 30, inclusive, four tags are drawn; required, the probability that 7 and 10 are among those drawn.

3. If, from a vase containing balls numbered 1, 2, 3, 4, 5, 6, it be required to obtain, in three drawings, first 1, then either 2 or 3, and, finally, 4, 5, or 6, what is the probability of doing so.

$$\text{Ans. } \frac{1}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{1}{20}.$$

4. In a box are 7 white balls, 5 red ones, and 3 black ones; what is the probability of obtaining, at a single drawing, 3 white balls, 2 red ones, and 1 black one?

The total number of possible chances is the number of different groups of $7+5+3=15$ balls, in sets of 6; this number is

$$\frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} = 5005. \text{ The 7 white balls}$$

may (544) be combined in sets of 3 in $\frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = 35$, different

ways; 35, then, is the whole number of groups of white balls, in sets of 3, that can be drawn. Similarly, $\frac{5 \cdot 4}{1 \cdot 2} = 10$, is the

whole number of groups of red balls in sets of 2; and as each group of white balls may be drawn with any one of the groups of red balls, the number of opportunities of drawing 3 white and 2 red balls is $35 \cdot 10 = 350$; and as one of the three black balls may be obtained in three ways, the whole number of opportunities of drawing 3 white balls, 2 red ones, and 1 black one, is $350 \cdot 3 = 1050$. Hence, the re-

$$\text{quired probability is } \frac{1050}{5005} = \frac{210}{1001} = \frac{30}{143}.$$

If 9 balls are drawn, what is the probability of there being 4 white, 3 red, and 2 black ones?

The whole number of chances is

$$\frac{9}{15} = \frac{15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = \frac{6}{15} = 5005.$$

The number of groups of white balls that may be drawn is

$$\frac{4}{7} = \frac{7 \cdot 6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{3}{7} = 35;$$

the number of groups of red balls is 10, and the number of groups of black balls, 2 in a group, is $\frac{{}_2C_2}{{}_3C_1} = \frac{3 \cdot 2}{1 \cdot 2} = 3$. As each group of black balls may be drawn with any one of the groups of red balls, and any one of the groups of red and black with any one of the groups of white, the whole number of favourable chances is $35 \cdot 10 \cdot 3 = 1050$, and the required probability is $\frac{1050}{5005} = \frac{30}{143}$.

5. From a bag containing 3 \$100 bills, 5 \$200 bills, and 8 \$10 bills, 8 bills are drawn; what is the probability that 3 \$10 bills, 2 \$200 bills, and 3 \$100 bills, will be found in those drawn?

6. On an average, 10 persons die in 10 years out of every 62 who are 30 years old at the beginning of the decade.

45 " 40 " " " " "

35 " 50 " " " " "

25 " 60 " " " " "

Required, the probability that a man, now 30, will live until he is 60.

The probability that he will live until he is 40 is $\frac{52}{62}$, for there are 62 chances in all, of which number 52 are favourable.

The probability of a man aged 40 living until he is 50 is $\frac{35}{45}$,

" " " 50 " " 60 is $\frac{25}{35}$,

for similar reasons. Hence, the required probability is $\frac{52}{62} \cdot \frac{35}{45} \cdot \frac{25}{35} = \frac{130}{279}$, since, in order to live until he is 60, the man must live through all three decades.

7. Required, the probability that a man, now 30, will live until he is 70; also, the probability that two persons, now 40, will live until they are 70.

8. Required, the probability that four persons, aged, respectively, 30, 40, 50, and 60, will all be alive at the end of 10 years.

CHAPTER XIII.

INTEREST.

556. Business calculations are often very much facilitated by the use of algebraic methods and formulæ. Some examples of this have already been given, and a few more, bearing on the subject of Interest, are here appended. While the multiplicity and variety of the problems that may arise precludes the possibility of an exhaustive treatment of the subject, it is hoped that the illustrations given, with the student's acquired skill, will enable him to apply, readily and accurately, his knowledge of algebra to the different problems of practical life.

The following notation will be used:

p denotes the number of dollars, pounds, francs, etc., bearing interest, *i. e.*, denotes the *principal*.

r denotes the fraction of 1 paid for the use of 1 (dollar, pound, franc, etc.), for one unit of time, as one year; it is usually a number of hundredths, and is called the *rate per cent*.

t denotes the number of units of time; we shall assume these units to be years, though quarters, months, etc., might be assumed equally well.

i denotes the interest of p for t years.

A denotes the amount of p for t years.

We shall assume the units of money to be dollars, unless otherwise specified.

557. To find the interest of p dollars for t years at the rate r .

$i = ptr$; for, since for the use of \$1 for a year r is paid, for \$ p for a year \$ pr must be paid, and since \$ pr is paid for

the use of \$ p for one year, for the use of \$ p for t years \$ p tr must be paid. Hence, $i = prt$.

558. To find the amount of p dollars for t years at the rate r .

$$A = p(1 + tr).$$

For, since the amount is the sum of the principal and the accrued interest, $A = p + i = p + ptr = p(1 + tr)$.

PRESENT VALUE AND DISCOUNT AT SIMPLE INTEREST.

559. To find the present value of A dollars due t years hence, r being the legal rate of interest.

$$P = \frac{A}{1 + tr}.$$

The present value of a sum due t years hence is the principal which, placed at interest for t years at the rate r , will produce the given sum. Hence, if p is this principal, A being the given amount, $A = p(1 + tr) \therefore p = \frac{A}{1 + tr}$.

560. To find the discount on a sum A , due t years hence, r being the legal rate.

$$\text{Let } D \text{ denote the discount required. Then } D = \frac{Atr}{1 + tr}.$$

For, discount being the difference between the sum due t years hence, and its present value,

$$D = A - \frac{A}{1 + tr}, \text{ or } D = \frac{A + Atr - A}{1 + tr} = \frac{Atr}{1 + tr}.$$

561. In the discussion of (559-60) it has been assumed that the debt does not bear interest. If A is to bear interest at the rate r' , $A(1 + tr')$ is the debt really due at the end of t years, and the A of the formula must, therefore, be replaced by $A(1 + tr')$.

562. As *four* quantities enter each of the foregoing formulæ, if *any three* of these quantities be given the *fourth* may be found by making the proper substitutions in the formulæ.

EXAMPLES.

1. Required, the interest and amount: of \$200 for 3 years, 10 months at 7%; of \$5000 for 9 months at 8%.

Interest. \$53.66+; \$300.

2. Required, the amount: of \$3050 for 4 years 8 months, at $5\frac{1}{2}\%$ per annum; of \$4562.09 for 3 years 3 months, at 3%; of \$1276.25 for 1 year 10 months, at 7%.

Ans. \$3797.25+; \$5006.89+; \$1440.03+.

3. What sum, invested in 7% bonds, at par, will yield an annual income of \$1260?

Ans. \$18000.

4. What is the interest for 11 years 8 days at $10\frac{1}{2}\%$, of a sum that, at the given rate, and in the given time, will amount to \$857.54.

Ans. \$460.04.

5. Show that, at simple interest, the discount is half the harmonic mean between the sum due and the interest thereon.

6. At what rate must \$1000 be loaned for three years, three months and twenty-nine days, to gain \$183.18.

Ans. $5\frac{1}{2}\%$.

7. Required, the time in which a given principal, p , will become n times itself, at the rate r .

Ans. $\frac{n-1}{r}$.

PARTIAL PAYMENTS.

563. To find what each payment must be, in order to discharge, in equal payments, made at equal intervals, the principal and interest of a given debt, bearing interest at a given rate.

We shall take one of the equal intervals as the unit of time, so that t will denote the number of these intervals in the whole time, and r will denote the interest on \$1 (or £, or fr., etc.), for one of these intervals; let d denote the number of dollars in each payment.

When the first interval has elapsed, the amount of the debt, p , is $p(1+r)$; the first payment is now made, leaving

a balance of $p(1+r)-d$, on which interest is to be calculated for the next interval, at the rate r . The amount due at the end of the second interval, is therefore

$$p(1+r)-d+pr(1+r)-dr, \text{ or } p(1+r)(1+r)-d(1+r), \\ \text{or } p(1+r)^2-d(1+r).$$

The second payment is now made, leaving a balance of $p(1+r)^2-d(1+r)-d$, on which, calculating the interest for the next interval, we obtain, as the amount due at the end of this time, $p(1+r)^3-d(1+r)^2-d(1+r)$, and subtracting the third payment, which is now made, we obtain

$$p(1+r)^3-d(1+r)^2-d(1+r)-d,$$

as the sum on which interest is calculated for the fourth interval, at the end of which time

$$p(1+r)^4-d(1+r)^3-d(1+r)^2-d(1+r)$$

is the amount due. Subtracting from this the fourth payment, and calculating the interest on the remainder, we find that the amount due at the end of the fifth interval is

$$p(1+r)^5-d(1+r)^4-d(1+r)^3-d(1+r)^2-d(1+r),$$

from which, deducting the fifth payment, we obtain

$$p(1+r)^5-d(1+r)^4-d(1+r)^3-d(1+r)^2-d(1+r)-d,$$

upon which interest is to be calculated for the sixth interval, and so on.

From the law of formation evident in these results, it follows that after t payments have been made there will remain $p(1+r)^t-d(1+r)^{t-1}-d(1+r)^{t-2}-\dots-d(1+r)^2-d(1+r)-d$. From the conditions of the problem, however, this must be zero, for the debt is completely discharged in t intervals. From the equation

$$p(1+r)^t-d(1+r)^{t-1}-d(1+r)^{t-2}-\dots-d(1+r)-d=0 \quad (1)$$

$$\text{we deduce } d = \frac{p(1+r)^t}{(1+r)^{t-1}+(1+r)^{t-2}+\dots+(1+r)+1} \quad (2)$$

The sum of the geometrical progression in the denominator is $\frac{(1+r)^t-1}{r}$. Substituting in (2), $d = \frac{p(1+r)^t}{\frac{(1+r)^t-1}{r}} = \frac{pr(1+r)^t}{(1+r)^t-1}$.

The foregoing solution, it will be seen, is based on the U. S. SUPREME COURT rule; a different solution is required

if any other rule be used. The solutions for the New Hampshire and Vermont rules are subjoined.

NEW HAMPSHIRE RULE.

Here let n = the number of equal intervals in a year, and t the number of years, r being the rate for each interval.

Then the amount of the principal p for 1 year is $p(1+nr)$; from which must be deducted the sum of the amounts of the payments from the times at which they are made to the end of the year. Let x denote one of the payments, then, as the first payment bears interest for $(n-1)$ equal intervals, the second for $(n-2)$, the third for $(n-3)$, etc.,

$x[1+(n-1)r]$ = amount of first payment at end of year.

$x[1+(n-2)r]$ = " " second " " " " "

$x[1+(n-3)r]$ = " " third " " " " "

$x[1+(n-n+1)r]$, or

$x[1+r]$ = amount of $(n-1)$ th " " " " "

x = " " n th " " " " "

The sum of these amounts is:

$x \{ [1+(n-1)r] + [1+(n-2)r] + \dots + [1+r] + 1 \}$, or

$x \{ n + r[(n-1) + (n-2) + (n-3) + \dots + 2 + 1] \}$

Summing the arithmetical progression in the brackets,

$S = \frac{(n-1)+1}{2}(n-1) = \frac{n(n-1)}{2}$; the whole quantity then

is $x \left\{ n + r \frac{n(n-1)}{2} \right\} = nx \left\{ 1 + r \frac{n-1}{2} \right\}$

This will be the same for every year, as each year the sum of the amounts of the payments for that year must be deducted from the amount of the principal for the year. For simplicity in carrying this quantity through the formulae, denote it by X , i. e., place $X = nx \left\{ 1 + r \frac{n-1}{2} \right\}$

Then the amount due at the end of the first year, after all payments for that year have been made, is $p(1+nr) - X$; which must be treated as a new principal, for the second year. The interest thereon, for this year is

$nr \{ p(1+nr) - X \}$,

and the amount, at the end of the year, is

$$p(1+nr)^2 - X(1+nr).$$

From this, deducting X , the amount of the payments for the year, we obtain $p(1+nr)^2 - X(1+nr) - X$, as the principal for the third year, the amount of which, at the end of the year, is $p(1+nr)^3 - X(1+nr)^2 - X(1+nr)$, and from this deducting X , the principal for the next, or fourth, year, is

$$p(1+nr)^3 - X(1+nr)^2 - X(1+nr) - X,$$

the amount of which, at the end of the fourth year, is

$$p(1+nr)^4 - X(1+nr)^3 - X(1+nr)^2 - X(1+nr) - X,$$

and the remainder, after deducting the amount of the payments, is $p(1+nr)^4 - X(1+nr)^3 - X(1+nr)^2 - X(1+nr) - X$. Finally, at the end of t years, the balance due will be $p(1+nr)^t - X(1+nr)^{t-1} - X(1+nr)^{t-2} - \dots - X(1+nr) - X$, which must be zero, as the debt is discharged in t years. From the equation,

$$p(1+nr)^t - X(1+nr)^{t-1} - \dots - X(1+nr) - X = 0, \quad (2),$$

we deduce, $X = \frac{prn(1+nr)^t}{(1+nr)^t - 1} = nx \left\{ 1 + r \frac{n-1}{2} \right\}$ Hence,

$$x = \frac{pr(1+nr)^t}{\{(1+nr)^t - 1\} \left\{ 1 + r \frac{n-1}{2} \right\}} \quad \text{or} \quad x = \frac{2pr(1+nr)^t}{[(1+nr)^t - 1][2 + r(n-1)]}$$

VERMONT RULE.

The formula for this rule may be deduced very simply from the preceding. As the interest is calculated on p for the whole time, instead of separate years, we may take t as 1, i. e., the whole time, letting n denote the number of equal intervals in this time. The last formula, then, becomes, substituting x' for x , $x' = \frac{2p(1+nr)}{n[2 + r(n-1)]}$.

Or, we may deduce x' directly from the expression for the first balance, which, as t is now taken as the unit, or 1, must be zero; thus, $p(1+nr) = X = nx' \left\{ \frac{2 + r(n-1)}{2} \right\} \therefore x' = \frac{2p(1+nr)}{n[2 + r(n-1)]}$.

EXAMPLES.

1. Borrowed \$3000 @ 7%, and repaid principal and interest in five equal annual payments. Required, the annual payment. U. S. S. Court rule, \$731.67. Find the payments required by the N. H. and Vt. rules.

2. A man purchased a farm for \$8500, and agreed to pay the sum in quarterly instalments, in four years, with interest at the rate of 10% per annum. Required, the annual payment.

3. What annual payment is required to discharge a note of \$32,000, bearing interest at 4% per annum, in six equal payments?

4. Purchased real estate for \$12,000; having but \$5000 on hand, used this sum to pay a part of the debt, and interest in advance, @ 10%, on a note running three years, for the balance. Required, the face of the note. *Ans.* \$10,000.

5. A man purchased a lot for \$1200, discharging the debt, principal and interest, at 4% per annum, in seven equal annual payments. Required, the annual payment.

COMPOUND INTEREST.

564. To find the amount of a sum p in t years, at the rate r , interest being compounded annually.

$$A = p(1+r)^t.$$

Since the interest is compounded annually, the amount of p for the first year is to be taken as a principal for the second year; the amount of this sum for the second year is to be taken as a principal for the third, and so on.

The amount of p for one year, at the rate r , (558) is $p(1+r)$; the amount of $p(1+r)$ for one year, at the rate r , is $p(1+r)(1+r) = p(1+r)^2$;
the amount of $p(1+r)^2$ for a year (the third year), at the rate

r , is $p(1+r)^2(1+r)=p(1+r)^3$, and, in general, the amount of $p(1+r)^{t-1}$ for a year being $p(1+r)^{t-1}(1+r)=p(1+r)^t$, the amount of p for t years at the rate r , interest being compounded annually, is $A=p(1+r)^t$.

565. Corollary.—*The present value of a sum p , due t years hence, is $\frac{A}{(1+r)^t}$, if compound interest be allowed. The discount, on the same hypothesis, is $A\left(1-\frac{1}{(1+r)^t}\right)$.*

566. If the interest is compounded quarterly, or at any other intervals than years, t denotes the number of such intervals.

567. The use of all the formulæ relating to interest is often facilitated by the employment of logarithms. To take, as a single example, the formula $A=p(1+r)^t$, we have $\log A=\log p+t\log(1+r)$. From this we may deduce at once $t=\frac{\log A-\log p}{\log(1+r)}$, or, in general, if any three of the quantities, p , A , r , or t , be given we may find the other.

EXAMPLES.

1. Required, the amount of \$7000 in five years, at 6%, interest being compounded quarterly.

2. Required, the amount of \$300 in four years, at 4%, interest compounded annually. *Ans.* \$350.96.

3. Required, the time in which a given principal will become n times itself, at compound interest, at the rate r .

$$\text{Ans. } t=\frac{\log n}{\log(1+r)}.$$

4. Required, the rate at which \$500 must be put out at compound interest, that it may, in seven years, amount to \$727.33. *Ans.* $5\frac{1}{2}\%$.

5. Required, the present value of the amount of £1000, due 5 years hence, at 5% per annum, compound interest.

ARITHMETICAL PROGRESSION.

TABLE I.—ARITHMETICAL PROGRESSION.

No.	Given.	Required.	Results.
1	a, δ, n	l, S	$l = a + (n-1)\delta; S = \frac{n[2a + (n-1)\delta]}{2}.$
2	l, δ, n	a, S	$a = l - (n-1)\delta; S = \frac{n[2l - (n-1)\delta]}{2}.$
3	a, l, n	δ, S	$\delta = \frac{l-a}{n-1}; S = \frac{n(a+l)}{2}.$
4	a, n, S	δ, l	$\delta = \frac{2(S-an)}{n(n-1)}; l = \frac{2S-an}{n}.$
5	n, δ, S	a, l	$a = \frac{2S-n(n-1)\delta}{2n}; l = \frac{2S+n(n-1)\delta}{2n}.$
6	l, n, S	a, δ	$a = \frac{2S-nl}{n}; \delta = \frac{2(nl-S)}{n(n-1)}.$
7	a, δ, l	n, S	$n = 1 + \frac{l-a}{\delta}; S = \frac{(l+a)(l-a+\delta)}{2\delta}.$
8	a, δ, S	l, n	$l = \frac{-\delta \pm \sqrt{8\delta S + (2a-\delta)^2}}{2}; n = \frac{\delta - 2a \pm \sqrt{8\delta S + (2a-\delta)^2}}{2\delta}.$
9	a, l, S	n, δ	$n = \frac{2S}{a+l}; \delta = \frac{2S-(a+l)}{2S-(a+l)}.$
10	l, δ, S	a, n	$a = \frac{\delta \pm \sqrt{(2l+\delta)^2 - 8\delta S}}{2}; n = \frac{2l+\delta \pm \sqrt{(2l+\delta)^2 - 8\delta S}}{2\delta}.$

GEOMETRICAL PROGRESSION.

TABLE II.—GEOMETRICAL PROGRESSION.

No.	Given.	Required.	Results.
1	a, ρ, n	l, S	$l = a\rho^{n-1}; S = \frac{a\rho^n - a}{\rho - 1} = \frac{a(\rho^n - 1)}{\rho - 1}.$
2	l, ρ, n	a, S	$a = \frac{l}{\rho^{n-1}}; S = \frac{l(\rho^n - 1)}{\rho^{n-1}(\rho - 1)}.$
3	n, ρ, S	a, l	$a = \frac{S(\rho - 1)}{\rho^{n-1}}; l = \frac{S\rho^{n-1}(\rho - 1)}{\rho^{n-1}}.$
4	a, l, n	ρ, S	$\rho = \sqrt[n]{\frac{l}{a}} = \left(\frac{l}{a}\right)^{\frac{1}{n-1}}; S = \frac{\frac{n-1}{\sqrt[n]{l}} - \frac{n-1}{\sqrt[n]{a}}}{\frac{n-1}{\sqrt[n]{l}} - \frac{n-1}{\sqrt[n]{a}}} = \frac{\frac{n}{l^{\frac{1}{n-1}}} - \frac{n}{a^{\frac{1}{n-1}}}}{\frac{n}{l^{\frac{1}{n-1}}} - \frac{n}{a^{\frac{1}{n-1}}}}.$
5	a, n, S	ρ, l	$\rho^{n-1} + \rho^{n-2} + \dots + 1 = \frac{S}{a}; l = a\rho^{n-1}; (S-l)^{n-1} = a(S-a)^{n-1}.$
6	l, n, S	a, ρ	$\left(\frac{1}{\rho}\right)^{n-1} + \left(\frac{1}{\rho}\right)^{n-2} + \dots + 1 = \frac{S}{l}; a = l\left(\frac{1}{\rho}\right)^{n-1}.$
7	a, ρ, l	S, n	$S = \frac{l\rho - a}{\rho - 1}; n = 1 + \frac{\log l - \log a}{\log \rho}.$
8	a, l, S	ρ, n	$\rho = \frac{S-a}{S-l}; n = 1 + \frac{\log(S-a) - \log(S-l)}{\log(a + S(\rho-1)) - \log a}.$
9	a, ρ, S	l, n	$l = \frac{a + S(\rho - 1)}{\rho}; n = \frac{\log \rho}{\log l - \log[a + S(\rho - 1)]}.$
10	l, ρ, S	a, n	$a = \rho l - (\rho - 1)S; n = 1 + \frac{\log l - \log[\rho l - (\rho - 1)S]}{\log \rho}.$

NOTE ON DETERMINANTS.

1. Let there be n^2 symbols, arranged in n rows and n columns, the sequence of the columns being indicated by the alphabetic order of the letters, and that of the rows by the numerical order of the subscripts, thus:—

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 & e_1 \\ a_2 & b_2 & c_2 & d_2 & e_2 \\ a_3 & b_3 & c_3 & d_3 & e_3 \\ a_4 & b_4 & c_4 & d_4 & e_4 \\ a_5 & b_5 & c_5 & d_5 & e_5 \end{vmatrix}$$

Let all possible products be formed by taking *one* symbol and *only* one from each column, and *one* symbol and *only* one from each row. When, in any such product, the letters being in alphabetical order, a suffix precedes one less than itself, we are said to have an *inversion*; if a suffix precede *two* smaller than itself we have *two* inversions, and so on. Give to each product the sign + or — according as the number of inversions in it is even or odd. The algebraic sum of these products is called the **Determinant** of the quantities involved. Each quantity is called a *constituent* (a , b , etc.), and each product, an *element*, of the determinant. The *order* of a determinant is indicated by the number of rows (or columns).

Thus the determinant of the second order, $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, is $a_1b_2 - a_2b_1$.
The determinant of the third order,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{cases} a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \\ -a_2b_1c_3 - a_3b_2c_1 - a_1b_3c_2 \end{cases}$$

and so on.

2. From the definition it is clear that if we change the *rows* into *columns*, or *vice versa*, the value of the determinant remains unchanged. Hence in all theorems the words “*row*” and “*column*” may be interchanged. The diagonal leading from the upper left corner of the square array is called the *leading diagonal*, and the product of its constituents the *leading element*. This element we always consider positive, and any other

element is positive or negative according as it may be derived from this by an even or an odd number of interchanges of subscripts.* The determinant is often expressed by prefixing to this element the double sign, $\pm$, and the summation sign, Σ ; or by simply inclosing this leading element in parentheses. Thus $\Sigma \pm a_1 b_2 c_3 = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$

$$= (a_1 b_2 c_3)$$

and similarly for determinants of higher orders.

3. As an interchange of two rows involves an interchange of the two corresponding suffixes in every element, and hence changes the sign of the element, such interchange of two rows changes the sign of the determinant. The effect of such interchange is always to increase or decrease by an odd number the number of inversions in any element. Thus, consider the term $a_1 b_2 c_3 d_4$ of the determinant $(a_1 b_2 c_3 d_4)$; the number of inversions is 2, and the term is therefore positive; interchange the third and fourth rows; 3 and 4 change places, and the number of inversions in $a_1 b_2 c_4 d_3$ is 3, so that this term is negative. Similarly for the other terms. Hence, also, an interchange of two columns changes the sign of the function. If two columns (or rows) be identical their interchange cannot affect the value of the determinant, which must therefore be equal to itself with sign changed. If, therefore, two columns be identical the determinant vanishes (i. e., equals 0).

From the definition it is also clear that if all the constituents of any column have a common factor, this quantity is a factor of the determinant. If, therefore, the constituents of any column be equal respectively to those of any other, multiplied by a common factor, the determinant must vanish.

4. *Minors*.—A *minor* is the determinant obtained by suppressing, in a given determinant, any number of rows and

*In fact the law of signs is often stated thus: "The leading element is taken as positive; any other element is positive or negative according as it is derived from the leading element by an even or an odd number of interchanges of suffixes."

the same number of columns, without changing the relative position of the constituents retained. The minors are called *first minors*, *second minors*, etc., according as we suppress *one row*, *two rows*, etc. Two minors are *complementary* when each is obtained by suppressing the rows and columns containing the constituents of the other.

Denoting the original determinant by Δ , consider the coefficient of any constituent, say a_1 . To obtain the elements of Δ that contain a_1 we must multiply a_1 by the sum of all the products obtained by taking one constituent and only one from each row and column in which a_1 does not occur. These are precisely the elements of the minor obtained by suppressing the row and column containing a_1 . Again, prefixing a_1 to any of these products cannot affect the number of inversions, so that the signs of the elements of Δ that contain a_1 are the same as those of the elements of $(b_1 c_3 d_4 \dots)$, the minor complementary to a_1 . Hence the coefficient of a_1 in Δ is this minor. Similarly a_2 must be multiplied by the successive elements of $(b_1 c_3 d_4 \dots)$, its complementary minor. But, since $2 > 1$ and $< 3, 4$, etc., prefixing a_2 to any element of this minor introduces one inversion into every term, and hence in each case a change of sign. The coefficient of a_2 is therefore $-(b_1 c_3 d_4 \dots)$. So also the coefficient of a_3 is $(b_1 c_2 d_4 \dots)$, that of a_4 is $-(b_1 c_2 d_3 \dots)$, and so on. If, then, we indicate the complementary minor of a_1 by A_1 , that of a_2 by A_2 , etc., we may write

$$\Delta = a_1 A_1 - a_2 A_2 + a_3 A_3 - a_4 A_4 + \dots \quad (1).$$

or, changing columns into rows, and *vice versa*

$$\Delta = a_1 A_1 - b_1 B_1 + c_1 C_1 - d_1 D_1 + \dots \quad (2).$$

In general, as a constituent in the r th column and the m th row can be brought to the upper left corner by r interchanges of columns, and m interchanges of rows, a process in which the sign of Δ is changed $(m+r)$ times, the coefficient of this constituent is $(-1)^{m+r}$ multiplied into its complementary minor, i. e., the minor obtained by suppressing the r th column and the m th row.

5. By reasoning similar to that of (4) we may show that the coefficient of *any minor* in Δ is the complementary minor, multiplied by $(-1)^n$; n denoting the number of interchanges of rows and columns required to bring the minor whose coefficient is sought into the upper left corner of the square array.

Hence Δ may be developed according to *minors* as well as in the manner just shown.

Thus $(a_1 b_1 c_1 d_1) =$

$$(a_1 b_2)(c_3 d_4) - (a_1 b_3)(c_2 d_4) + (a_1 b_4)(c_2 d_3) \\ + (a_2 b_1)(c_3 d_4) - (a_2 b_3)(c_1 d_4) + (a_2 b_4)(c_1 d_3)$$

6. From equations (1), (2), of (4) we see that if every constituent except one in any column (or row) vanishes, the determinant reduces to $(-1)^n$ into the complementary minor of that one; n denoting the number of interchanges of columns and rows required to bring the non-vanishing constituent into the leading place (that usually occupied by a_1). Thus, if all the constituents of the first column vanish except a_r , $\Delta = -a_r A_r$, and similarly for any other case.

7. Again, if every constituent in any one column be the sum of two (or m) quantities the determinant may be treated as the sum of two (or m) determinants.

Thus, if $a_1 = k_1 + l_1$, $a_2 = k_2 + l_2$, $a_3 = k_3 + l_3$ etc.,

$$\Delta = a_1 A_1 - a_2 A_2 + a_3 A_3 - \dots = \\ (k_1 + l_1) A_1 - (k_2 + l_2) A_2 + (k_3 + l_3) A_3 - \dots = \\ k_1 A_1 - k_2 A_2 + k_3 A_3 - \dots + l_1 A_1 - l_2 A_2 + l_3 A_3 - \dots \\ = (k_1 b_1 c_1 d_1 \dots) + (l_1 b_1 c_1 d_1 \dots). \quad (3).$$

8. The value of a determinant is not affected by the addition to the constituents of any column of the products of the respective constituents of any other column by any common multiplier. Thus

$$\begin{vmatrix} a_1 + kc_1 & b_1 & c_1 \\ a_2 + kc_2 & b_2 & c_2 \\ a_3 + kc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} kc_1 & b_1 & c_1 \\ kc_2 & b_2 & c_2 \\ kc_3 & b_3 & c_3 \end{vmatrix} \text{ by (7).}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + k \begin{vmatrix} c_1 & b_1 & c_1 \\ c_1 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ by (3).}$$

Similarly for determinants of higher orders.

Neither is the value of a determinant altered if to the constituents of any column we add the respective constituents of two or more other columns multiplied in each case by a fixed multiplier; thus it is easily shown that

$$\begin{vmatrix} a_1 & b_1 + ka_1 + la_1 & c_1 \\ a_2 & b_2 + ka_2 + la_2 & c_2 \\ a_3 & b_3 + ka_3 + la_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

9. The foregoing transformations are used in reducing the degree of a determinant, to simplify its evaluation. Thus:—

$$\begin{vmatrix} 3 & 5 & 6 \\ 6 & 11 & 3 \\ 2 & 4 & 9 \end{vmatrix} = 3 \begin{vmatrix} 3 & 5 & 2 \\ 6 & 11 & 1 \\ 2 & 4 & 3 \end{vmatrix} = 3 \begin{vmatrix} -9 & -17 & 0 \\ 6 & 11 & 1 \\ -16 & -29 & 0 \end{vmatrix} = -3 \begin{vmatrix} 9 & 17 \\ 16 & 29 \end{vmatrix},$$

$$= -3(261 - 272) = 33.$$

The second determinant is obtained by taking out the factor 3 from the last column of the first. The second row is then multiplied through by 2 and 3 successively and the results are subtracted from the first and third rows respectively. Finally we apply (3) and expand the resulting determinant of the second order.

10. Determinants are very useful in the solution of problems involving elimination.

We may write the identities

$$b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4 + \dots = 0 \quad (\alpha)$$

$$a_1A_1 - c_2A_2 + c_3A_3 - c_4A_4 + \dots = 0 \quad (\beta)$$

$$d_1A_1 - d_2A_2 + d_3A_3 - d_4A_4 + \dots = 0 \quad (\gamma)$$

For, the first member of (α) is what Δ becomes when $b = a$ (Eq. 1, (4)). But in this case the first and second columns are identical, whence $\Delta = 0$, (3). Similarly the first member of (β) is what Δ becomes when the first and third columns are identical, and so on.

Now suppose we wish to solve the group of equations

$$a_1x + a_2y + a_3z = k_1 \quad (1)$$

$$b_1x + b_2y + b_3z = k_2 \quad (2)$$

$$c_1x + c_2y + c_3z = k_3 \quad (3)$$

Multiply both members of (1) by A_1 , both members of (2) by $-A_2$, both members of (3) by A_3 , and add. The coefficients of y and z vanish by virtue of identities (α) and (β), and we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix} \text{ or, } x = \frac{(k_1 \ b_2 \ c_3)}{(a_1 \ b_2 \ c_3)}.$$

$$\text{Similarly, } y = \frac{(a_1 \ k_2 \ c_3)}{(a_1 \ b_2 \ c_3)}; z = \frac{(a_1 \ b_2 \ k_3)}{(a_1 \ b_2 \ c_3)}.$$

These values have as a common denominator Δ , the determinant of the coefficients in the first members of the given equations. To obtain the numerator of the value of any variable we substitute, in Δ , the second members of the equations for the column of coefficients of that variable. Ordinarily the values may be left in the determinant form. If, however, evaluation be required it may be effected as in (9). When it is desired simply to *eliminate* the variables or unknown constants the result is best expressed in determinant form. Thus, the result of eliminating a, b, c , from the equations

$$\begin{aligned} ax' + by' + c &= 0, \\ ax'' + by'' + c &= 0, \\ ax''' + by''' + c &= 0, \end{aligned} \text{ is } \begin{vmatrix} x' & y' & 1 \\ x'' & y'' & 1 \\ x''' & y''' & 1 \end{vmatrix} = 0$$

EXERCISES.

1. Evaluate the determinants

$$\begin{vmatrix} 3 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{vmatrix}, \begin{vmatrix} 1 & a-b \\ -a & 1 & c \\ b-c & 1 \end{vmatrix}, \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \begin{vmatrix} 5 & -2 & 1 \\ 32 & 7 & 1 \\ 1 & 1 & 8 \end{vmatrix}$$

Ans., $1^\circ, 360$; $2^\circ, 1 + a^2 + b^2 + c^2$; $4^\circ, 810$.

2. Evaluate the determinants

$$\begin{vmatrix} 1 & 15 & 14 & 4 \\ 12 & 6 & 7 & 9 \\ 8 & 10 & 11 & 5 \\ 13 & 3 & 2 & 16 \end{vmatrix}, \begin{vmatrix} 10 & 18 & 1 & 14 & 22 \\ 4 & 12 & 25 & 8 & 16 \\ 23 & 6 & 19 & 2 & 15 \\ 17 & 5 & 13 & 21 & 9 \\ 11 & 24 & 7 & 20 & 3 \end{vmatrix}$$

The first is composed of the first sixteen numbers, and the second of the first twenty-five numbers, so arranged that the sum of the constituents in any full straight line of the determinant equals the sum of those in any other, no matter how taken. Such arrays are called "Magic Squares."

Ans., 0; -4680000.

3. Evaluate the determinants

$$\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}, \begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}, \begin{vmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{vmatrix}$$

4. Solve the groups of equations in (373,376) by use of determinants.

5. Show that

$$\begin{vmatrix} a_1k_1+b_1l_1+c_1m_1 & a_1k_2+b_1l_2+c_1m_2 & a_1k_3+b_1l_3+c_1m_3 \\ a_2k_1+b_2l_1+c_2m_1 & a_2k_2+b_2l_2+c_2m_2 & a_2k_3+b_2l_3+c_2m_3 \\ a_3k_1+b_3l_1+c_3m_1 & a_3k_2+b_3l_2+c_3m_2 & a_3k_3+b_3l_3+c_3m_3 \end{vmatrix}$$

is the product of the two determinants $(a_1b_1c_1)$ and $(k_1l_1m_1)$, and that, in general,

The product of two determinants of the same order is the determinant the constituents of which are the sums of the products obtained by multiplying the constituents of any row (or column) of the one by the corresponding constituents in the other.

6. Two constituents are said to be conjugate to each other when the position occupied by either in the rows is the same as that occupied by the other in the columns. When, in any determinant, the conjugate constituents are equal, the determinant is said to be *symmetrical*. Show that the square of a symmetrical determinant is a symmetrical determinant.

7. Show that the square of the third determinant of Ex. 3 is

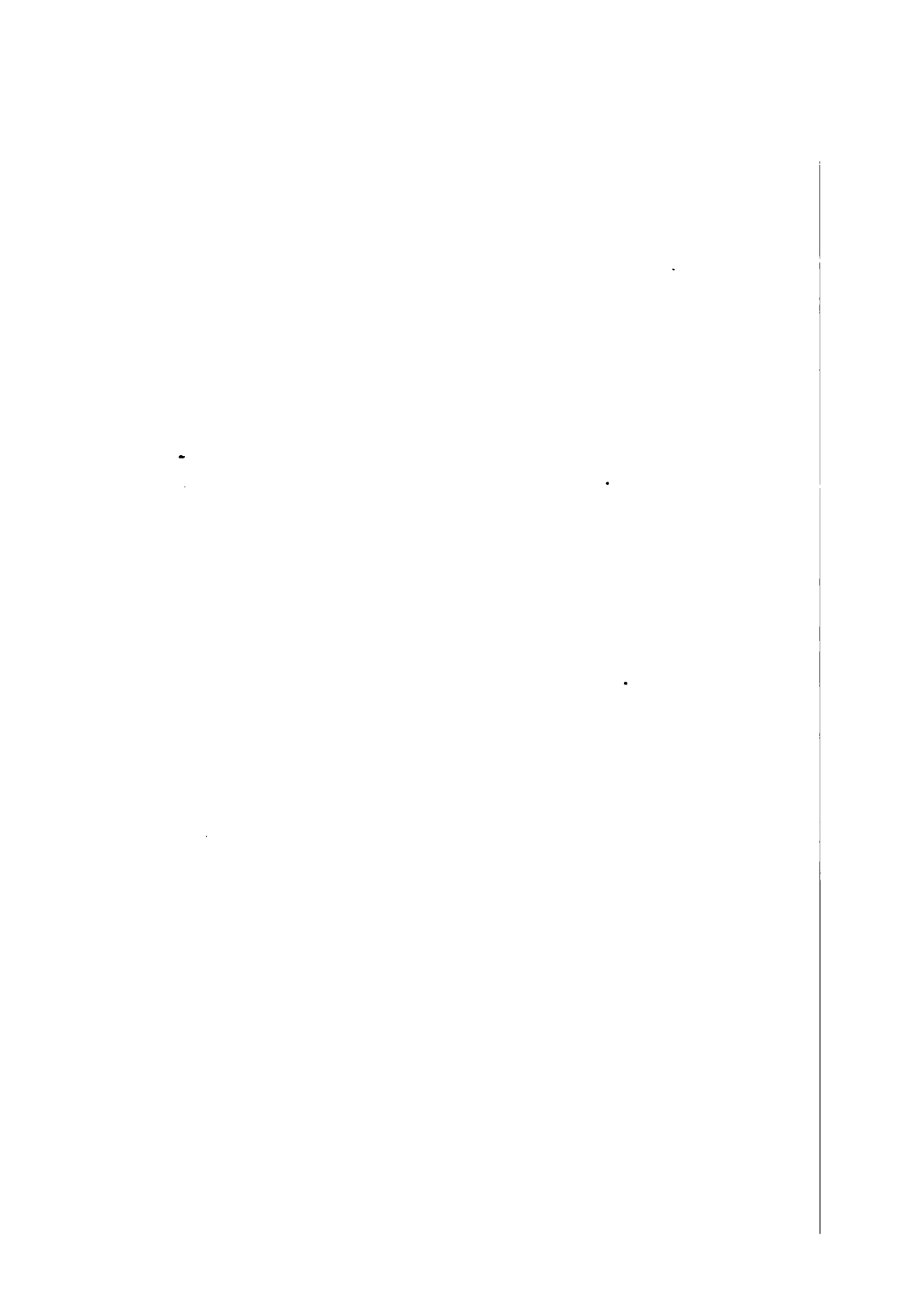
$$\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}$$

where

$$s_1 = a + b + c + d,$$

$$s_2 = a^2 + b^2 + c^2 + d^2,$$

and so on.



III.

TABLE OF LOGARITHMS.

1. To find, from the table, the logarithm of any number.

I. If the number is expressed by less than three figures, find the number in one of the columns headed *N*, on the first page; immediately to the right will be found the logarithm required. Thus $\log 73 = 1.863823$.

II. If the number is expressed by three figures, or more, find the characteristic by (528).

1°. If the number is expressed by three figures, find it in one of the columns headed *N*; immediately to the right will be the mantissa, or the last four figures thereof; in the latter case ascend the column until a row of six figures is found, and prefix the first two of these to the four figures previously found.

2°. If the number is expressed by four figures, find the number expressed by its first three figures in the column *N*; on the same horizontal line, in the column headed by the fourth figure, will be found the last four figures of the mantissa, dots, if any be found, being replaced by 0's. If heavy dots have been met in passing across to the last four figures of the mantissa, take the first two figures on the next lower horizontal line in the 0 column, and prefix them to the four first found; the result is the complete mantissa. If no heavy dots have been met, and the first three figures of the number stand opposite a row of six figures in the 0 column, the first two figures of this row are the ones to be prefixed to the four figures of the mantissa previously found. If a

row of six figures does not stand opposite the first three figures of the number, ascend the 0 column until such a row is found, and prefix the first two figures thereof to the four mantissa figures already found.

$$\begin{aligned}\text{Thus } \log 1205 &= 3.080987, \\ \log 1324 &= 3.121888, \\ \log 1853 &= 3.267875.\end{aligned}$$

3°. If the given number exceeds 10,000, place a decimal point after the fourth figure, thus converting the number into a mixed number (528). Find the mantissa of the integral part; take from the column D the number on the same horizontal line as the first three figures of the given number, and, multiplying this, which is always millionths, by the decimal part of the given number as pointed off, add the product to the mantissa of the integral part; the result will be the mantissa required. The numbers in the column D, called **Tabular Differences**, are the differences between the logarithms of consecutive whole numbers, at the different parts of the table. They are found by subtracting the number in the 4 column from the corresponding number in the 5 column.

Take the number 1357625. The characteristic of the logarithm of this number is 6. Placing a decimal point between the 7 and the 6 will not affect the mantissa of the logarithm (529); this decimal point being placed, the number becomes 1357.625. The mantissa of $\log 1357$ is .132580; the tabular difference at this point is 321, i. e., a difference of 1 between two numbers connotes a difference of 321 millionths between their logarithms. Since an increase of 1 in a number at this point implies an increase of .000321 in the mantissa of the logarithm, an increase of $\frac{625}{1000}$ of 1 implies an increase of $\frac{625}{1000}$ of .000321 in the mantissa (nearly). $\frac{625}{1000}$ of .000321 = .000201. Hence the mantissa of $\log 1357.625$ = mantissa of $\log 1357$ + .000201 = .132781. Therefore, $\log 1357625 = 6.132781$, the mantissa of $\log 1357625$ being identical with the mantissa of $\log 1357.625$. Similarly, $\log 79635279 = 7.901106$.

2. To find, from the table, the logarithm of a fraction or mixed number.

Reduce the given number to a decimal or mixed decimal; find the characteristic by (528); and, dropping the decimal point, find the mantissa as indicated in (1).

3. To find the number corresponding to a given logarithm.

To do this, the processes just indicated must be reversed. Look for the given mantissa in the table, seeking the first two figures in the 0 column; the others may be in the columns to the right. If the mantissa is found, the last figure of the required number will head the column containing the last four figures of the mantissa, and the first three figures of the required number will be found on the same horizontal line as the last part of the mantissa, and in the N column. The numbers corresponding to logarithms on the first page may also be obtained directly from this page.

If the mantissa is not found in the table, take out the next less mantissa, and set aside the corresponding number. Divide the difference between the mantissa of the given logarithm and that taken out, by the tabular difference corresponding to the latter, annexing as many 0's as may be necessitated by the required degree of accuracy. Annex the quotient to the number set aside, and point off the number of places corresponding to the characteristic of the given logarithm. To illustrate, find the number corresponding to the logarithm 1.662758; this logarithm, being given on the first page of the table, we take out at once the corresponding number, 46. The number corresponding to the logarithm 3.528145, is 3374. In searching for the number whose logarithm is 2.449918, however, we cannot find the mantissa in the table; the next less mantissa is .449787, and the corresponding number 2817, and the tabular difference 154. The difference between .449918 and .449787 is .000131. Now, as a difference of .000154 in the

mantissas implies a difference of 1 in the corresponding numbers, a difference of $.000131 = \frac{131}{154}$, of $.000154$ implies a difference of $\frac{131}{154}$ of 1 in the numbers,

$$\frac{131}{154} = \frac{154}{1232} \quad \frac{1310}{1232} \quad | \frac{851}{100} = .851.$$

The increase of $.000131$ in the mantissa involving an increase of $.851+$ in the number, the number corresponding to the mantissa $.449918$, is 2817.851 , or (529) , any number having these significant figures. Hence, the significant figures of the required number are 2817851 , and since the characteristic is 2, the required number is $281.7851+$.

To write out the arithmetical complement of a logarithm, begin at the left and subtract each figure from 9, until the last significant figure is reached, which subtract from 10.

A TABLE
OF
LOGARITHMS OF NUMBERS
FROM 1 TO 10,000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.
1	0.000000	26	1.414973	51	1.707570	76	1.880814
2	0.301030	27	1.431364	52	1.716003	77	1.886491
3	0.477121	28	1.447158	53	1.724276	78	1.892095
4	0.602060	29	1.462398	54	1.732394	79	1.897627
5	0.698970	30	1.477121	55	1.740363	80	1.903090
6	0.778151	31	1.491362	56	1.748188	81	1.908485
7	0.845098	32	1.505150	57	1.755875	82	1.913814
8	0.903090	33	1.518514	58	1.763428	83	1.919078
9	0.954243	34	1.531479	59	1.770852	84	1.924279
10	1.000000	35	1.544068	60	1.778151	85	1.929419
11	1.041393	36	1.556303	61	1.785330	86	1.934498
12	1.079181	37	1.568202	62	1.792392	87	1.939519
13	1.113943	38	1.579784	63	1.799341	88	1.944453
14	1.146128	39	1.591065	64	1.806180	89	1.949390
15	1.176091	40	1.602060	65	1.812913	90	1.954243
16	1.204120	41	1.612784	66	1.819544	91	1.959041
17	1.230449	42	1.623249	67	1.826075	92	1.963788
18	1.255273	43	1.633468	68	1.832509	93	1.968483
19	1.278754	44	1.643453	69	1.838849	94	1.973128
20	1.301030	45	1.653213	70	1.845098	95	1.977724
21	1.322219	46	1.662758	71	1.851258	96	1.982271
22	1.342423	47	1.672098	72	1.857333	97	1.986772
23	1.361728	48	1.681241	73	1.863323	98	1.991226
24	1.380211	49	1.690196	74	1.869232	99	1.995635
25	1.397940	50	1.698970	75	1.875061	100	2.000000

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	*300	*724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	*361	*775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	*195	*600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	*207	*602	*998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	389
112	9218	9606	9993	*380	*766	1153	1538	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	*320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	*38	*407	*776	1145	1514	369
118	071882	2260	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	*266	*626	*987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	*258	*611	*963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	*26	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	*253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	*245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	*12	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	*194	*508	*822	1136	1450	1763	2076	2389	2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	*142	*449	*756	1063	1370	1676	1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	*168	*469	*769	1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
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A TABLE OF LOGARITHMS FROM 1 TO 10,000.

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150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	•126	•413	•699	•985	1272	1558	287
152	131844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	••51	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	••29	•303	•577	•850	1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	••51	•319	•586	•853	1121	1388	1654	1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	•193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	••50	•300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	•176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	6•32	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	•213	•446	•679	•912	1144	1377	1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4158	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	8982	9211	9439	9667	9895	•123	•351	•578	•806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	•161	•378	•595	•813	218
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200	301030	1247	1464	1681	1898	2114	2331	2547	2764	2980	217
201	3196	3412	3628	3844	4059	4275	4491	4706	4921	5136	216
202	5351	5566	5781	5996	6211	6425	6639	6854	7068	7282	215
203	7496	7710	7924	8137	8351	8564	8778	8991	9204	9417	213
204	9630	9843	*56	*268	*481	*693	*906	1118	1330	1542	212
205	311754	1966	2177	2389	2600	2812	3023	3234	3445	3656	211
206	3867	4078	4289	4499	4710	4920	5130	5340	5551	5760	210
207	5970	6180	6390	6599	6809	7018	7227	7436	7646	7854	209
208	8063	8272	8481	8689	8898	9106	9314	9522	9730	9938	208
209	320146	0354	0562	0769	0977	1184	1391	1598	1805	2012	207
210	322219	2426	2633	2839	3046	3252	3458	3665	3871	4077	206
211	4282	4488	4694	4899	5105	5310	5516	5721	5926	6131	205
212	6336	6541	6745	6950	7155	7359	7563	7767	7972	8176	204
213	8380	8583	8787	8991	9194	9398	9601	9805	***8	*211	203
214	330414	0617	0819	1022	1225	1427	1630	1832	2034	2236	202
215	2438	2640	2842	3044	3246	3447	3649	3850	4051	4253	202
216	4454	4655	4856	5057	5257	5458	5658	5859	6059	6260	201
217	6460	6660	6860	7060	7260	7459	7659	7858	8058	8257	200
218	8456	8656	8855	9054	9253	9451	9650	9849	**47	*246	199
219	340444	0642	0841	1039	1237	1435	1632	1830	2028	2225	198
220	342423	2620	2817	3014	3212	3409	3606	3802	3999	4196	197
221	4392	4589	4785	4981	5178	5374	5570	5766	5962	6157	196
222	6353	6549	6744	6939	7135	7330	7525	7720	7915	8110	195
223	8305	8500	8694	8889	9083	9278	9472	9666	9860	**54	194
224	350248	0442	0636	0829	1023	1216	1410	1603	1796	1989	193
225	2183	2375	2568	2761	2954	3147	3339	3532	3724	3916	193
226	4108	4301	4493	4685	4876	5068	5260	5452	5643	5834	192
227	6026	6217	6408	6599	6790	6981	7172	7363	7554	7744	191
228	7935	8125	8316	8506	8696	8886	9076	9266	9456	9646	190
229	9835	*25	*215	*404	*593	*783	*972	1161	1350	1539	189
230	361728	1917	2105	2294	2482	2671	2859	3048	3236	3424	188
231	3612	3800	3988	4176	4363	4551	4739	4926	5113	5301	188
232	5488	5675	5862	6049	6236	6423	6610	6796	6983	7169	187
233	7356	7542	7729	7915	8101	8287	8473	8659	8845	9030	186
234	9216	9401	9587	9772	9958	*143	*328	*513	*698	*883	185
235	371068	1253	1437	1622	1806	1991	2175	2360	2544	2728	184
236	2912	3096	3280	3464	3647	3831	4015	4198	4382	4565	184
237	4748	4932	5115	5298	5481	5664	5846	6029	6212	6394	183
238	6577	6759	6942	7124	7306	7488	7670	7852	8034	8216	182
239	8398	8580	8761	8943	9124	9306	9487	9668	9849	**30	181
240	380211	0392	0573	0754	0934	1115	1296	1476	1656	1837	181
241	2017	2197	2377	2557	2737	2917	3097	3277	3456	3636	180
242	3815	3995	4174	4353	4533	4712	4891	5070	5249	5428	179
243	5606	5785	5964	6142	6321	6499	6677	6856	7034	7212	178
244	7390	7568	7746	7923	8101	8279	8456	8634	8811	8989	178
245	9166	9343	9520	9698	9875	**51	*228	*405	*582	*759	177
246	390935	1112	1288	1464	1641	1817	1993	2169	2345	2521	176
247	2697	2873	3045	3224	3400	3575	3751	3926	4101	4277	176
248	4452	4627	4802	4977	5152	5326	5501	5676	5850	6025	175
249	6199	6374	6548	6722	6896	7071	7245	7419	7592	7766	174
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250	397940	8114	8287	8461	8634	8808	8981	9154	9328	9501	173
251	9674	9847	••20	•192	•365	•538	•711	•883	1056	1228	1.8
252	401401	1573	1745	1917	2089	2261	2433	2605	2777	2949	17.2
253	3121	3292	3464	3635	3807	3978	4149	4320	4492	4663	171
254	4834	5005	5176	5346	5517	5688	5858	6029	6199	6370	171
255	6540	6710	6881	7051	7221	7391	7561	7731	7901	8070	170
256	8240	8410	8579	8749	8918	9087	9257	9426	9595	9764	169
257	9933	•102	•271	•440	•609	•777	•946	1114	1283	1451	169
258	411620	1788	1956	2124	2293	2461	2629	2796	2964	3132	168
259	3300	3467	3635	3803	3970	4137	4305	4472	4639	4806	167
260	414973	5140	5307	5474	5641	5808	5974	6141	6308	6474	167
261	6641	6807	6973	7139	7306	7472	7638	7804	7970	8135	166
262	8301	8467	8633	8798	8964	9129	9295	9460	9625	9791	165
263	9956	•121	•286	•451	•616	•781	•945	1110	1275	1439	165
264	421601	1768	1933	2097	2261	2426	2590	2754	2918	3082	164
265	3246	3410	3574	3737	3901	4065	4228	4392	4555	4718	164
266	4882	5045	5208	5371	5534	5697	5860	6023	6186	6349	163
267	6511	6674	6836	6999	7161	7324	7486	7648	7811	7973	162
268	8135	8297	8459	8621	8783	8944	9106	9268	9429	9591	162
269	9752	9914	••75	•236	•398	•559	•720	•881	1042	1203	161
270	431364	1525	1685	1846	2007	2167	2328	2488	2649	2809	161
271	2969	3130	3290	3450	3610	3770	3930	4090	4249	4409	160
272	4569	4729	4888	5048	5207	5367	5526	5685	5844	6004	159
273	6163	6322	6481	6640	6799	6957	7115	7275	7433	7592	159
274	7751	7909	8067	8226	8384	8542	8701	8859	9017	9175	158
275	9333	9491	9648	9806	9964	•122	•279	•437	•594	•752	158
276	440909	1066	1224	1381	1538	1695	1852	2009	2166	2323	157
277	2480	2637	2793	2950	3106	3263	3419	3576	3732	3889	157
278	4045	4201	4357	4513	4669	4825	4981	5137	5293	5449	156
279	5604	5760	5915	6071	6226	6382	6537	6692	6848	7003	155
280	447158	7313	7468	7623	7778	7933	8088	8242	8397	8552	155
281	8706	8861	9015	9170	9324	9478	9633	9787	9941	••95	154
282	450249	0403	0557	0711	0865	1018	1172	1326	1479	1633	154
283	1786	1940	2093	2247	2400	2553	2706	2859	3012	3165	153
284	3318	3471	3624	3777	3930	4082	4235	4387	4540	4692	152
285	4845	4997	5150	5302	5454	5606	5758	5910	6062	6214	152
286	6366	6518	6670	6821	6973	7125	7276	7428	7579	7731	152
287	7882	8033	8184	8336	8487	8638	8789	8940	9091	9242	151
288	9392	9543	9694	9845	9995	•146	•296	•447	•597	•748	151
289	460898	1048	1198	1348	1499	1649	1799	1948	2098	2248	150
290	462398	2548	2697	2847	2997	3146	3296	3445	3594	3744	150
291	3893	4042	4191	4340	4490	4639	4788	4936	5085	5234	149
292	5383	5532	5680	5829	5977	6126	6274	6423	6571	6719	149
293	6868	7016	7164	7312	7460	7608	7756	7904	8052	8200	148
294	8347	8495	8643	8790	8938	9085	9233	9380	9527	9675	148
295	9822	9969	•116	•263	•410	•557	•704	•851	•998	1145	147
296	471291	1438	1585	1732	1878	2025	2171	2318	2464	2610	146
297	2756	2903	3049	3195	3341	3487	3633	3779	3925	4071	146
298	4216	4362	4508	4653	4799	4944	5090	5235	5381	5526	146
299	5671	5816	5962	6107	6252	6397	6542	6687	6832	6976	145
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301	8566	8711	8855	8999	9143	9287	9431	9575	9719	9863	144
302	480007	0151	0294	0438	0582	0725	0869	1012	1156	1299	144
303	1443	1586	1729	1872	2016	2159	2302	2445	2588	2731	143
304	2874	3016	3159	3302	3445	3587	3730	3872	4015	4157	143
305	4300	4442	4585	4727	4869	5011	5153	5295	5437	5579	142
306	5721	5863	6005	6147	6289	6430	6572	6714	6855	6997	142
307	7138	7280	7421	7563	7704	7845	7986	8127	8269	8410	141
308	8551	8692	8833	8974	9114	9255	9396	9537	9677	9818	141
309	9958	**99	*239	*380	*520	*661	*801	*941	1081	1222	140
310	491362	1502	1642	1782	1922	2062	2201	2341	2481	2621	140
311	2760	2900	3040	3179	3319	3458	3597	3737	3876	4015	139
312	4155	4291	4433	4572	4711	4850	4989	5128	5267	5406	139
313	5544	5683	5822	5960	6099	6238	6376	6515	6653	6791	139
314	6930	7068	7206	7344	7483	7621	7759	7897	8035	8173	138
315	8311	8448	8586	8724	8862	8999	9137	9275	9412	9550	138
316	9687	9824	9962	**99	*236	*374	*511	*648	*785	*922	137
317	501059	1196	1333	1470	1607	1744	1880	2017	2154	2291	137
318	2427	2564	2700	2837	2973	3109	3246	3382	3518	3655	136
319	3791	3927	4063	4199	4335	4471	4607	4743	4878	5014	136
320	505150	5286	5421	5557	5693	5828	5964	6099	6234	6370	136
321	6505	6640	6776	6911	7046	7181	7316	7451	7586	7721	135
322	7856	7991	8126	8260	8395	8530	8664	8799	8934	9068	135
323	9203	9337	9471	9606	9740	9874	**9	*143	*277	*411	134
324	510545	0679	0813	0947	1081	1215	1349	1482	1616	1750	134
325	1883	2017	2151	2284	2418	2551	2684	2818	2951	3084	133
326	3218	3351	3484	3617	3750	3883	4016	4149	4282	4414	133
327	4548	4681	4813	4946	5079	5211	5344	5476	5609	5741	133
328	5874	6006	6139	6271	6403	6535	6668	6800	6932	7064	132
329	7196	7328	7460	7592	7724	7855	7987	8119	8251	8383	132
330	518514	8646	8777	8909	9040	9171	9303	9434	9566	9697	131
331	9828	9959	**90	*221	*353	*484	*615	*745	*876	1007	131
332	521138	1269	1400	1530	1661	1792	1922	2053	2183	2314	131
333	2444	2575	2705	2835	2966	3096	3226	3356	3486	3616	130
334	3746	3876	4006	4136	4266	4396	4526	4656	4785	4915	130
335	5045	5174	5304	5434	5563	5693	5822	5951	6081	6210	129
336	6339	6469	6598	6727	6856	6985	7114	7243	7372	7501	129
337	7630	7759	7888	8016	8145	8274	8402	8531	8660	8789	129
338	8917	9045	9174	9302	9430	9559	9687	9815	9943	**72	128
339	530200	0328	0456	0584	0712	0840	0968	1096	1223	1351	128
340	531479	1607	1734	1862	1990	2117	2245	2372	2500	2627	128
341	2754	2882	3009	3136	3264	3391	3518	3645	3772	3899	127
342	4026	4153	4280	4407	4534	4661	4787	4914	5041	5167	127
343	5294	5421	5547	5674	5800	5927	6053	6180	6306	6432	126
344	6558	6685	6811	6937	7063	7189	7315	7441	7567	7693	126
345	7819	7945	8071	8197	8322	8448	8574	8699	8825	8951	126
346	9076	9202	9327	9452	9578	9703	9829	9954	**79	*204	125
347	540329	0455	0580	0705	0830	0955	1080	1205	1330	1454	125
348	1579	1704	1829	1953	2078	2203	2327	2452	2576	2701	125
349	2825	2950	3074	3199	3323	3447	3571	3696	3820	3944	124
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A TABLE OF LOGARITHMS FROM 1 TO 10,000.

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351	5307	5431	5555	5678	5802	5925	6049	6172	6296	6419	124
352	6543	6666	6789	6913	7036	7159	7282	7405	7529	7652	123
353	7775	7898	8021	8144	8267	8389	8512	8635	8758	8881	123
354	9003	9126	9249	9371	9494	9616	9739	9861	9984	*106	123
355	550228	0351	0473	0595	0717	0840	0962	1084	1206	1328	122
356	1450	1572	1694	1816	1938	2060	2181	2303	2425	2547	122
357	2668	2790	2911	3033	3155	3276	3398	3519	3640	3762	121
358	3883	4004	4126	4247	4368	4489	4610	4731	4852	4973	121
359	5094	5215	5336	5457	5578	5699	5820	5940	6061	6182	121
360	556303	6423	6544	6664	6785	6905	7026	7146	7267	7387	120
361	7507	7627	7748	7868	7988	8108	8228	8349	8469	8589	120
362	8709	8829	8948	9068	9188	9308	9428	9548	9667	9787	120
363	9907	**26	*146	*265	*385	*504	*624	*743	*863	*982	119
364	561101	1221	1340	1459	1578	1698	1817	1936	2055	2174	119
365	2293	2412	2531	2650	2769	2887	3006	3125	3244	3362	119
366	3481	3600	3718	3837	3955	4074	4192	4311	4429	4548	119
367	4666	4784	4903	5021	5139	5257	5376	5494	5612	5730	118
368	5348	5966	6084	6202	6320	6437	6555	6673	6791	6909	118
369	7026	7144	7262	7379	7497	7614	7732	7849	7967	8084	118
370	568202	8319	8436	8554	8671	8788	8905	9023	9140	9257	117
371	9374	9491	9608	9725	9842	9959	**76	*193	*309	*426	117
372	570543	0660	0776	0893	1010	1126	1243	1359	1476	1592	117
373	1709	1825	1942	2058	2174	2291	2407	2523	2639	2755	116
374	2872	2988	3104	3220	3336	3452	3568	3684	3800	3915	116
375	4031	4147	4263	4379	4494	4610	4726	4841	4957	5072	116
376	5188	5303	5419	5534	5650	5765	5880	5996	6111	6226	115
377	6341	6457	6572	6687	6802	6917	7032	7147	7262	7377	115
378	7492	7607	7722	7836	7951	8066	8181	8295	8410	8525	115
379	8539	8754	8868	8983	9097	9212	9326	9441	9555	9669	114
380	579784	9898	**12	*126	*241	*355	*469	*583	*697	*811	114
381	580925	1039	1153	1267	1381	1495	1608	1722	1836	1950	114
382	2063	2177	2291	2404	2518	2631	2745	2858	2972	3085	114
383	3199	3312	3426	3539	3652	3765	3879	3992	4105	4218	113
384	4331	4444	4557	4670	4783	4896	5009	5122	5235	5348	113
385	5461	5574	5686	5799	5912	6024	6137	6250	6362	6475	113
386	6587	6700	6812	6925	7037	7149	7262	7374	7486	7599	112
387	7711	7823	7935	8047	8160	8272	8384	8496	8608	8720	112
388	8832	8944	9056	9167	9279	9391	9503	9615	9726	9838	112
389	9950	**61	*173	*284	*396	*507	*619	*730	*842	*953	112
390	591065	1176	1287	1399	1510	1621	1732	1843	1955	2066	111
391	2177	2288	2399	2510	2621	2732	2843	2954	3064	3175	111
392	3286	3397	3508	3618	3729	3840	3950	4061	4171	4282	111
393	4393	4503	4614	4724	4834	4945	5055	5165	5276	5386	110
394	5496	5606	5717	5827	5937	6047	6157	6267	6377	6487	110
395	6597	6707	6817	6927	7037	7146	7256	7366	7476	7586	110
396	7695	7805	7914	8024	8134	8243	8353	8462	8572	8681	110
397	8791	8900	9009	9119	9228	9337	9446	9556	9665	9774	109
398	9883	9992	*101	*210	*319	*428	*537	*646	*755	*864	109
399	600973	1082	1191	1299	1408	1517	1625	1734	1843	1951	109
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400	602060	2169	2277	2386	2494	2603	2711	2819	2928	3036	108
401	3144	3253	3361	3469	3577	3686	3794	3902	4010	4118	108
402	4226	4334	4442	4550	4658	4766	4874	4982	5089	5197	108
403	5305	5413	5521	5628	5736	5844	5951	6059	6166	6274	108
404	6381	6489	6596	6704	6811	6919	7026	7133	7241	7348	107
405	7455	7562	7669	7777	7884	7991	8098	8205	8312	8419	107
406	8526	8633	8740	8847	8954	9061	9167	9274	9381	9488	107
407	9594	9701	9808	9914	*21	*128	*234	*341	*447	*554	107
408	610660	0767	0873	0979	1086	1192	1298	1405	1511	1617	106
409	1723	1829	1936	2042	2148	2254	2360	2466	2572	2678	106
410	612784	2890	2996	3102	3207	3313	3419	3525	3630	3736	106
411	3842	3947	4053	4159	4264	4370	4475	4581	4686	4791	106
412	4897	5003	5108	5213	5319	5424	5529	5634	5740	5845	105
413	5950	6055	6160	6265	6370	6476	6581	6686	6790	6895	105
414	7000	7105	7210	7315	7420	7525	7629	7734	7839	7943	105
415	8048	8153	8257	8362	8466	8571	8676	8780	8884	8989	105
416	9093	9198	9302	9406	9511	9615	9719	9824	9928	**32	104
417	620136	0240	0344	0448	0552	0656	0760	0864	0968	1072	104
418	1176	1280	1384	1488	1592	1695	1799	1903	2007	2110	104
419	2214	2318	2421	2525	2628	2732	2835	2939	3042	3146	104
420	623249	3353	3456	3559	3663	3766	3869	3973	4076	4179	103
421	4282	4385	4488	4591	4695	4798	4901	5004	5107	5210	103
422	5312	5415	5518	5621	5724	5827	5929	6032	6135	6238	103
423	6340	6443	6546	6648	6751	6853	6956	7058	7161	7263	103
424	7366	7468	7571	7673	7775	7878	7980	8082	8185	8287	102
425	8389	8491	8593	8695	8797	8900	9002	9104	9206	9308	102
426	9410	9512	9613	9715	9817	9919	*21	*123	*224	*326	102
427	630428	0530	0631	0733	0835	0936	1038	1139	1241	1342	102
428	1444	1545	1647	1748	1849	1951	2052	2153	2255	2356	101
429	2457	2559	2660	2761	2862	2963	3064	3165	3266	3367	101
430	633468	3569	3670	3771	3872	3973	4074	4175	4276	4376	100
431	4477	4578	4679	4779	4880	4981	5081	5182	5283	5383	100
432	5484	5584	5685	5785	5886	5986	6087	6187	6287	6388	100
433	6488	6588	6688	6789	6889	6989	7089	7189	7290	7390	100
434	7490	7590	7690	7790	7890	7990	8090	8190	8290	8389	99
435	8489	8589	8689	8789	8888	8988	9088	9188	9287	9387	99
436	9486	9586	9686	9785	9885	9984	*84	*183	*283	*382	99
437	640481	0581	0680	0779	0879	0978	1077	1177	1276	1375	99
438	1474	1573	1672	1771	1871	1970	2069	2168	2267	2366	99
439	2465	2563	2662	2761	2860	2959	3058	3156	3255	3354	99
440	643453	3551	3650	3749	3847	3946	4044	4143	4242	4340	98
441	4439	4537	4636	4734	4832	4931	5029	5127	5226	5324	98
442	5422	5521	5619	5717	5815	5913	6011	6110	6208	6306	98
443	6404	6502	6600	6698	6796	6894	6992	7089	7187	7285	98
444	7383	7481	7579	7676	7774	7872	7969	8067	8165	8262	98
445	8360	8458	8555	8653	8750	8848	8945	9043	9140	9237	97
446	9335	9432	9530	9627	9724	9821	9919	*16	*113	*210	97
447	650308	0405	0502	0599	0696	0793	0890	0987	1084	1181	97
448	1278	1375	1472	1569	1666	1762	1859	1956	2053	2150	97
449	2246	2343	2440	2536	2633	2730	2826	2923	3019	3116	97
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450	653213	3909	3405	3502	3598	3695	3791	3888	3984	4080	96
451	4177	4273	4369	4465	4562	4658	4754	4850	4946	5042	96
452	5138	5235	5331	5427	5523	5619	5715	5810	5906	6002	96
453	6098	6194	6290	6386	6482	6577	6673	6769	6864	6960	96
454	7056	7152	7247	7343	7438	7534	7629	7725	7820	7916	96
455	8011	8107	8202	8298	8393	8488	8584	8679	8774	8870	95
456	8965	9060	9155	9250	9346	9441	9536	9631	9726	9821	95
457	9916	*11	*106	*201	*296	*391	*486	*581	*676	*771	95
458	660865	0960	1055	1150	1245	1339	1434	1529	1623	1718	95
459	1813	1907	2002	2096	2191	2286	2380	2475	2569	2663	95
460	662758	2852	2947	3041	3135	3230	3324	3418	3512	3607	94
461	3701	3795	3889	3983	4078	4172	4266	4360	4454	4548	94
462	4642	4736	4830	4924	5018	5112	5206	5299	5393	5487	94
463	5551	5645	5739	5832	5926	6020	6113	6207	6301	6394	94
464	6518	6612	6705	6799	6892	6986	7079	7173	7266	7360	94
465	7453	7546	7640	7733	7826	7920	8013	8106	8199	8293	93
466	8386	8479	8572	8665	8759	8852	8945	9038	9131	9224	93
467	9317	9410	9503	9596	9689	9782	9875	9967	*60	*153	93
468	670246	0339	0431	0524	0617	0710	0802	0895	0988	1080	93
469	1173	1265	1358	1451	1543	1636	1728	1821	1913	2005	93
470	672098	2190	2283	2375	2467	2560	2652	2744	2836	2929	92
471	3021	3113	3205	3297	3390	3482	3574	3666	3758	3850	92
472	3942	4034	4126	4218	4310	4402	4494	4586	4677	4769	92
473	4861	4953	5045	5137	5228	5320	5412	5503	5595	5687	92
474	5778	5870	5962	6053	6145	6236	6328	6419	6511	6602	92
475	6694	6785	6876	6968	7059	7151	7242	7333	7424	7516	91
476	7607	7698	7789	7881	7972	8063	8154	8245	8336	8427	91
477	8518	8609	8700	8791	8882	8973	9064	9155	9246	9337	91
478	9428	9519	9610	9700	9791	9882	9973	*63	*154	*245	91
479	680336	0426	0517	0607	0698	0789	0879	0970	1060	1151	91
480	681241	1332	1422	1513	1603	1693	1784	1874	1964	2055	90
481	2145	2235	2326	2416	2506	2596	2686	2777	2867	2957	90
482	3047	3137	3227	3317	3407	3497	3587	3677	3767	3857	90
483	3947	4037	4127	4217	4307	4396	4486	4576	4666	4756	90
484	4845	4935	5025	5114	5204	5294	5383	5473	5563	5652	90
485	5742	5831	5921	6010	6100	6189	6279	6368	6458	6547	89
486	6636	6726	6815	6904	6994	7083	7172	7261	7351	7440	89
487	7529	7618	7707	7796	7886	7975	8064	8153	8242	8331	89
488	8420	8509	8598	8687	8776	8865	8953	9042	9131	9220	89
489	9309	9398	9486	9575	9664	9753	9841	9930	*19	*107	89
490	690196	0285	0373	0462	0550	0639	0728	0816	0905	0993	89
491	1081	1170	1258	1347	1435	1524	1612	1700	1789	1877	88
492	1965	2053	2142	2230	2318	2406	2494	2583	2671	2759	88
493	2847	2935	3023	3111	3199	3287	3375	3463	3551	3639	88
494	3727	3815	3903	3991	4078	4166	4254	4342	4430	4517	88
495	4605	4693	4781	4868	4956	5044	5131	5219	5307	5394	88
496	5482	5569	5657	5744	5832	5919	6007	6094	6182	6269	87
497	6356	6444	6531	6618	6706	6793	6880	6968	7055	7142	87
498	7229	7317	7404	7491	7578	7665	7752	7839	7926	8014	87
499	8101	8188	8275	8362	8449	8535	8622	8709	8796	8883	87
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501	9838	9924	••11	••98	•184	•271	•358	•444	•531	•617	87
502	700704	0790	0577	0963	1050	1136	1222	1309	1395	1482	86
503	1568	1654	1741	1827	1913	1999	2086	2172	2258	2344	86
504	2431	2517	2603	2689	2775	2861	2947	3033	3119	3205	86
505	3291	3377	3463	3549	3635	3721	3807	3895	3979	4065	86
506	4151	4236	4322	4408	4494	4579	4665	4751	4837	4922	86
507	5008	5094	5179	5265	5350	5436	5522	5607	5693	5778	86
508	5864	5949	6035	6120	6206	6291	6376	6462	6547	6632	85
509	6718	6803	6888	6974	7059	7144	7229	7315	7400	7485	85
510	707570	7655	7740	7826	7911	7996	8081	8166	8251	8336	85
511	8421	8506	8591	8676	8761	8846	8931	9015	9100	9185	85
512	9270	9355	9440	9524	9609	9694	9779	9863	9948	••33	85
513	710117	0203	0287	0371	0456	0540	0625	0710	0794	0879	85
514	0963	1048	1132	1217	1301	1385	1470	1554	1639	1723	84
515	1807	1892	1976	2060	2144	2229	2313	2397	2481	2566	84
516	2650	2734	2818	2902	2986	3070	3154	3238	3323	3407	84
517	3491	3575	3659	3742	3826	3910	3994	4078	4162	4246	84
518	4320	4414	4497	4581	4665	4749	4833	4916	5000	5084	84
519	5167	5251	5335	5418	5502	5586	5669	5753	5836	5920	84
520	716003	6087	6170	6254	6337	6421	6504	6588	6671	6754	83
521	6838	6921	7004	7088	7171	7254	7338	7421	7504	7587	83
522	7671	7754	7837	7920	8003	8086	8169	8253	8336	8419	83
523	8502	8585	8668	8751	8834	8917	9000	9083	9165	9248	83
524	9331	9414	9497	9580	9663	9745	9828	9911	9994	••77	83
525	720159	0242	0325	0407	0490	0573	0655	0738	0821	0903	83
526	0986	1068	1151	1233	1316	1398	1481	1563	1646	1728	82
527	1811	1893	1975	2058	2140	2222	2305	2387	2469	2552	82
528	2634	2716	2798	2881	2963	3045	3127	3209	3291	3374	82
529	3456	3538	3620	3702	3784	3866	3948	4030	4112	4194	82
530	724276	4358	4440	4522	4604	4685	4767	4849	4931	5013	82
531	5095	5176	5258	5340	5422	5503	5585	5667	5748	5830	82
532	5912	5993	6075	6156	6238	6320	6401	6483	6564	6646	82
533	6727	6809	6890	6972	7053	7134	7216	7297	7379	7460	81
534	7541	7623	7704	7785	7866	7948	8029	8110	8191	8273	81
535	8354	8435	8516	8597	8678	8759	8841	8922	9003	9084	81
536	9165	9246	9327	9408	9489	9570	9651	9732	9813	9893	81
537	9974	••55	•136	•217	•298	•378	•459	•540	•621	•702	81
538	730782	0863	0944	1024	1105	1186	1266	1347	1428	1508	81
539	1589	1669	1750	1830	1911	1991	2072	2152	2233	2313	81
540	732394	2474	2555	2635	2715	2796	2876	2956	3037	3117	80
541	3197	3278	3358	3438	3518	3598	3679	3759	3839	3919	80
542	3999	4079	4160	4240	4320	4400	4480	4560	4640	4720	80
543	4800	4880	4960	5040	5120	5200	5279	5359	5439	5519	80
544	5599	5679	5759	5838	5918	5998	6078	6157	6237	6317	80
545	6397	6476	6556	6635	6715	6795	6874	6954	7034	7113	80
546	7193	7272	7352	7431	7511	7590	7670	7749	7829	7908	79
547	7987	8067	8146	8225	8305	8384	8463	8543	8622	8701	79
548	8781	8860	8939	9018	9097	9177	9256	9335	9414	9493	79
549	9572	9651	9731	9810	9889	9968	••47	•126	•205	•284	79
N.	0	1	2	3	4	5	6	7	8	9	D.

A TABLE OF LOGARITHMS FROM 1 TO 10,000.

xv

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550	740363	0442	0521	0600	0678	0757	0836	0915	0994	1073	79
551	1152	1230	1309	1388	1467	1546	1624	1703	1782	1860	79
552	1939	2018	2096	2175	2254	2332	2411	2489	2568	2646	79
553	2725	2804	2882	2961	3039	3118	3196	3275	3353	3431	78
554	3510	3588	3667	3745	3823	3902	3980	4058	4136	4215	78
555	4293	4371	4449	4528	4606	4684	4762	4840	4919	4997	78
556	5075	5153	5231	5309	5387	5465	5543	5621	5699	5777	78
557	5855	5933	6011	6089	6167	6245	6323	6401	6479	6556	78
558	6634	6712	6790	6868	6945	7023	7101	7179	7256	7334	78
559	7412	7489	7567	7645	7722	7800	7878	7955	8033	8110	78
560	748188	8266	8343	8421	8498	8576	8653	8731	8808	8885	77
561	8963	9040	9118	9195	9272	9350	9427	9504	9582	9659	77
562	9736	9814	9891	9968	*45	*123	*200	*277	*354	*431	77
563	750508	0586	0663	0740	0817	0894	0971	1048	1125	1202	77
564	1279	1356	1433	1510	1587	1664	1741	1818	1895	1972	77
565	2048	2125	2202	2279	2356	2433	2509	2586	2663	2740	77
566	2816	2893	2970	3047	3123	3200	3277	3353	3430	3506	77
567	3583	3660	3736	3813	3889	3966	4042	4119	4195	4272	77
568	4348	4425	4501	4578	4654	4730	4807	4883	4960	5036	76
569	5112	5189	5265	5341	5417	5494	5570	5646	5722	5799	76
570	755875	5951	6027	6103	6180	6256	6332	6408	6484	6560	76
571	6636	6712	6788	6864	6940	7016	7092	7168	7244	7320	76
572	7396	7472	7548	7624	7700	7775	7851	7927	8003	8079	76
573	8155	8230	8306	8382	8458	8533	8609	8685	8761	8836	76
574	8912	8988	9063	9139	9214	9290	9366	9441	9517	9592	76
575	9668	9743	9819	9894	9970	*45	*121	*196	*272	*347	75
576	760422	0498	0573	0649	0724	0799	0875	0950	1025	1101	75
577	1176	1251	1326	1402	1477	1552	1627	1702	1778	1853	75
578	1928	2003	2078	2153	2228	2303	2378	2453	2529	2604	75
579	2679	2754	2829	2904	2978	3053	3128	3203	3278	3353	75
580	763428	3503	3578	3653	3727	3802	3877	3952	4027	4101	75
581	4176	4251	4326	4400	4475	4550	4624	4699	4774	4848	75
582	4923	4998	5072	5147	5221	5296	5370	5445	5520	5594	75
583	5669	5743	5818	5892	5966	6041	6115	6190	6264	6338	74
584	6413	6487	6562	6636	6710	6785	6859	6933	7007	7082	74
585	7156	7230	7304	7379	7453	7527	7601	7675	7749	7823	74
586	7898	7972	8046	8120	8194	8268	8342	8416	8490	8564	74
587	8638	8712	8786	8860	8934	9008	9082	9156	9230	9303	74
588	9377	9451	9525	9599	9673	9746	9820	9894	9968	*42	74
589	770115	0189	0263	0336	0410	0484	0557	0631	0705	0778	74
590	770852	0926	0999	1073	1146	1220	1293	1367	1440	1514	74
591	1587	1661	1734	1808	1881	1955	2028	2102	2175	2248	73
592	2322	2395	2468	2542	2615	2688	2762	2835	2908	2981	73
593	3055	3128	3201	3274	3348	3421	3494	3567	3640	3713	73
594	3786	3860	3933	4006	4079	4152	4225	4298	4371	4444	73
595	4517	4590	4663	4736	4809	4882	4955	5028	5100	5173	73
596	5246	5319	5392	5465	5538	5610	5683	5756	5829	5902	73
597	5974	6047	6120	6193	6265	6338	6411	6483	6556	6629	73
598	6701	6774	6846	6919	6992	7064	7137	7209	7282	7354	73
599	7427	7499	7572	7644	7717	7789	7862	7934	8006	8079	72
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600	778151	8224	8296	8368	8441	8513	8585	8658	8730	8802	72
601	8874	8947	9019	9091	9163	9236	9308	9380	9452	9524	72
602	9596	9669	9741	9813	9885	9957	••29	•101	•173	•245	72
603	780317	0389	0461	0533	0605	0677	0749	0821	0893	0965	72
604	1037	1109	1181	1253	1324	1396	1468	1540	1612	1684	72
605	1755	1827	1899	1971	2042	2114	2186	2258	2329	2401	72
606	2473	2544	2616	2688	2759	2831	2902	2974	3046	3117	72
607	3189	3260	3332	3403	3475	3546	3618	3689	3761	3832	71
608	3904	3975	4046	4118	4189	4261	4332	4403	4475	4546	71
609	4617	4689	4760	4831	4902	4974	5045	5116	5187	5259	71
610	785330	5401	5472	5543	5615	5686	5757	5828	5899	5970	71
611	6041	6112	6183	6254	6325	6396	6467	6538	6609	6680	71
612	6751	6822	6893	6964	7035	7106	7177	7248	7319	7390	71
613	7460	7531	7602	7673	7744	7815	7885	7956	8027	8098	71
614	8168	8239	8310	8381	8451	8522	8593	8663	8734	8804	71
615	8875	8946	9016	9087	9157	9228	9299	9369	9440	9510	71
616	9581	9651	9722	9792	9863	9933	•••4	••74	•144	•215	70
617	790285	0356	0426	0496	0567	0637	0707	0778	0848	0918	70
618	0938	1059	1129	1199	1269	1340	1410	1480	1550	1620	70
619	1691	1761	1831	1901	1971	2041	2111	2181	2252	2322	70
620	792392	2462	2532	2602	2672	2742	2812	2882	2952	3022	70
621	3092	3162	3231	3301	3371	3441	3511	3581	3651	3721	70
622	3790	3860	3930	4000	4070	4139	4209	4279	4349	4418	70
623	4488	4558	4627	4697	4767	4836	4906	4976	5045	5115	70
624	5185	5254	5324	5393	5463	5532	5602	5672	5741	5811	70
625	5880	5949	6019	6088	6158	6227	6297	6366	6436	6505	69
626	6574	6644	6713	6782	6852	6921	6990	7060	7129	7198	69
627	7268	7337	7406	7475	7545	7614	7683	7752	7821	7890	69
628	7960	8029	8098	8167	8236	8305	8374	8443	8513	8582	69
629	8651	8720	8789	8858	8927	8996	9065	9134	9203	9272	69
630	799341	9409	9478	9547	9616	9685	9754	9823	9892	9961	69
631	800029	0098	0167	0236	0305	0373	0442	0511	0580	0648	69
632	0717	0786	0854	0923	0992	1061	1129	1198	1266	1335	69
633	1404	1472	1541	1609	1678	1747	1815	1884	1952	2021	69
634	2089	2158	2226	2295	2363	2432	2500	2568	2637	2705	69
635	2774	2842	2910	2979	3047	3116	3184	3252	3321	3389	68
636	3457	3525	3594	3662	3730	3798	3867	3935	4003	4071	68
637	4139	4208	4276	4344	4412	4480	4548	4616	4685	4753	68
638	4821	4889	4957	5025	5093	5161	5229	5297	5365	5433	68
639	5501	5569	5637	5705	5773	5841	5908	5976	6044	6112	68
640	806180	6248	6316	6384	6451	6519	6587	6655	6723	6790	68
641	6858	6926	6994	7061	7129	7197	7264	7332	7400	7467	68
642	7535	7603	7670	7738	7806	7873	7941	8008	8076	8143	68
643	8211	8279	8346	8414	8481	8549	8616	8684	8751	8818	67
644	8886	8953	9021	9088	9156	9223	9290	9358	9425	9492	67
645	9560	9627	9694	9762	9829	9896	9964	••31	••98	•165	67
646	810233	0300	0367	0434	0501	0569	0636	0703	0770	0837	67
647	0904	0971	1039	1106	1173	1240	1307	1374	1441	1508	67
648	1575	1642	1709	1776	1843	1910	1977	2044	2111	2178	67
649	2245	2312	2379	2445	2512	2579	2646	2713	2780	2847	67
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650	812913	2980	3047	3114	3181	3247	3314	3381	3448	3514	67
651	3581	3648	3714	3781	3848	3914	3981	4048	4114	4181	67
652	4248	4314	4381	4447	4514	4581	4647	4714	4780	4847	67
653	4913	4980	5046	5113	5179	5246	5312	5378	5445	5511	66
654	5578	5644	5711	5777	5843	5910	5976	6042	6109	6175	66
655	6241	6308	6374	6440	6506	6573	6639	6705	6771	6838	66
656	6904	6970	7036	7102	7169	7235	7301	7367	7433	7499	66
657	7565	7631	7698	7764	7830	7896	7962	8028	8094	8160	66
658	8226	8292	8358	8424	8490	8556	8622	8688	8754	8820	66
659	8885	8951	9017	9083	9149	9215	9281	9346	9412	9478	66
660	819544	9610	9676	9741	9807	9873	9939	••••	••70	•136	66
661	820201	0267	0333	0399	0464	0530	0595	0661	0727	0792	66
662	0858	0924	0989	1055	1120	1186	1251	1317	1382	1448	66
663	1514	1579	1645	1710	1775	1841	1906	1972	2037	2103	65
664	2168	2233	2299	2364	2430	2495	2560	2626	2691	2756	65
665	2822	2887	2952	3018	3083	3148	3213	3279	3344	3409	65
666	3474	3539	3605	3670	3735	3800	3865	3930	3996	4061	65
667	4126	4191	4256	4321	4386	4451	4516	4581	4646	4711	65
668	4776	4841	4906	4971	5036	5101	5166	5231	5296	5361	65
669	5426	5491	5556	5621	5686	5751	5815	5880	5945	6010	65
670	826075	6140	6204	6269	6334	6399	6464	6528	6593	6658	65
671	6723	6787	6852	6917	6981	7046	7111	7175	7240	7305	65
672	7369	7434	7499	7563	7628	7692	7757	7821	7886	7951	65
673	8015	8080	8144	8209	8273	8338	8402	8467	8531	8595	64
674	8660	8724	8789	8853	8918	8982	9046	9111	9175	9239	64
675	9304	9368	9432	9497	9561	9625	9690	9754	9818	9882	64
676	9947	••11	••75	•139	•204	•268	•332	•396	•460	•525	64
677	830589	0653	0717	0781	0845	0909	0973	1037	1102	1166	64
678	1230	1294	1358	1422	1486	1550	1614	1678	1742	1806	64
679	1870	1934	1998	2062	2126	2189	2253	2317	2381	2445	64
680	832509	2573	2637	2700	2764	2828	2892	2950	3020	3083	64
681	3147	3211	3275	3338	3402	3466	3530	3593	3657	3721	64
682	3784	3848	3912	3975	4039	4103	4166	4230	4294	4357	64
683	4421	4484	4548	4611	4675	4739	4802	4866	4929	4993	64
684	5056	5120	5183	5247	5310	5373	5437	5500	5564	5627	63
685	5691	5754	5817	5881	5944	6007	6071	6134	6197	6261	63
686	6324	6387	6451	6514	6577	6641	6704	6767	6830	6894	63
687	6957	7020	7083	7146	7210	7273	7336	7399	7462	7525	63
688	7558	7652	7715	7778	7841	7904	7967	8030	8093	8156	63
689	8219	8282	8345	8408	8471	8534	8597	8660	8723	8786	63
690	838849	8912	8975	9038	9101	9164	9227	9289	9352	9415	63
691	9478	9541	9604	9667	9729	9792	9855	9918	9981	••43	63
692	840106	0169	0232	0294	0357	0420	0482	0545	0608	0671	63
693	0733	0796	0859	0921	0984	1046	1109	1172	1234	1297	63
694	1359	1422	1485	1547	1610	1672	1735	1797	1860	1922	63
695	1985	2047	2110	2172	2235	2297	2360	2422	2484	2547	62
696	2609	2672	2734	2796	2859	2921	2983	3046	3108	3170	62
697	3233	3295	3357	3420	3482	3544	3606	3669	3731	3793	62
698	3855	3918	3980	4042	4104	4166	4229	4291	4353	4415	62
699	4477	4539	4601	4664	4726	4788	4850	4912	4974	5036	62
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xviii A TABLE OF LOGARITHMS FROM 1 TO 10,000.

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700	845098	5160	5222	5284	5346	5408	5470	5532	5594	5656	62
701	5718	5780	5842	5904	5966	6028	6090	6151	6213	6275	62
702	6337	6399	6461	6523	6585	6646	6708	6770	6832	6894	62
703	6955	7017	7079	7141	7202	7264	7326	7388	7449	7511	62
704	7573	7634	7696	7758	7819	7881	7942	8004	8066	8128	62
705	8189	8251	8312	8374	8435	8497	8559	8620	8682	8743	62
706	8805	8866	8928	8989	9051	9112	9174	9235	9297	9358	61
707	9419	9481	9542	9604	9665	9726	9788	9849	9911	9972	61
708	850033	0095	0156	0217	0279	0340	0401	0462	0524	0585	61
709	0646	0707	0769	0830	0891	0952	1014	1075	1136	1197	61
710	851258	1320	1381	1442	1503	1564	1625	1686	1747	1809	61
711	1870	1931	1992	2053	2114	2.75	2236	2297	2358	2419	61
712	2480	2541	2602	2663	2724	2785	2846	2907	2968	3029	61
713	3090	3150	3211	3272	3333	3394	3455	3516	3577	3637	61
714	3698	3759	3820	3881	3941	4002	4063	4124	4185	4245	61
715	4306	4367	4428	4488	4549	4610	4670	4731	4792	4852	61
716	4913	4974	5034	5095	5156	5216	5277	5337	5398	5459	61
717	5519	5580	5640	5701	5761	5822	5882	5943	6003	6064	61
718	6124	6185	6245	6306	6366	6427	6487	6548	6608	6668	60
719	6729	6789	6850	6910	6970	7031	7091	7152	7212	7272	60
720	857332	7393	7453	7513	7574	7634	7694	7755	7815	7875	60
721	7935	7995	8056	8116	8176	8236	8297	8357	8417	8477	60
722	8537	8597	8657	8718	8778	8838	8898	8958	9018	9078	60
723	9138	9198	9258	9318	9379	9439	9499	9559	9619	9679	60
724	9739	9799	9859	9918	9978	••38	••98	•158	•218	•278	60
725	860338	0398	0458	0518	0578	0637	0697	0757	0817	0877	60
726	0937	0996	1056	1116	1176	1236	1295	1355	1415	1475	60
727	1534	1594	1654	1714	1773	1833	1893	1952	2012	2072	60
728	2131	2191	2251	2310	2370	2430	2489	2549	2608	2668	60
729	2728	2787	2847	2906	2966	3025	3085	3144	3204	3263	60
730	863323	3382	3442	3501	3561	3620	3680	3739	3799	3858	59
731	3917	3977	4036	4096	4155	4214	4274	4333	4392	4452	59
732	4511	4570	4630	4689	4748	4808	4867	4926	4985	5045	59
733	5104	5163	5222	5282	5341	5400	5459	5519	5578	5637	59
734	5696	5755	5814	5874	5933	5992	6051	6110	6169	6228	59
735	6287	6346	6405	6465	6524	6583	6642	6701	6760	6819	59
736	6878	6937	6996	7055	7114	7173	7232	7291	7350	7409	59
737	7467	7526	7585	7644	7703	7762	7821	7880	7939	7998	59
738	8056	8115	8174	8233	8292	8350	8409	8468	8527	8586	59
739	8644	8703	8762	8821	8879	8938	8997	9056	9114	9173	59
740	869232	9290	9349	9408	9466	9525	9584	9642	9701	9760	59
741	9818	9877	9935	9994	••53	•111	•170	•228	•287	•345	59
742	870404	0462	0521	0579	0638	0696	0755	0813	0872	0930	58
743	0989	1047	1106	1164	1223	1281	1339	1398	1456	1515	58
744	1573	1631	1690	1748	1806	1865	1923	1981	2040	2098	58
745	2156	2215	2273	2331	2389	2448	2506	2564	2622	2681	58
746	2739	2797	2855	2913	2972	3030	3088	3146	3204	3262	58
747	3321	3379	3437	3495	3553	3611	3669	3727	3785	3844	58
748	3902	3960	4018	4076	4134	4192	4250	4308	4366	4424	58
749	4482	4540	4598	4656	4714	4772	4830	4888	4945	5003	58
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750	875061	5119	5177	5235	5293	5351	5409	5466	5524	5582	58
751	5640	5698	5756	5813	5871	5929	5987	6045	6102	6160	58
752	6218	6276	6333	6391	6449	6507	6564	6622	6680	6737	58
753	6795	6853	6910	6968	7026	7083	7141	7199	7256	7314	58
754	7371	7429	7487	7544	7602	7659	7717	7774	7832	7889	58
755	7947	8004	8062	8119	8177	8234	8292	8349	8407	8464	57
756	8522	8579	8637	8694	8752	8809	8866	8924	8981	9039	57
757	9096	9153	9211	9268	9325	9383	9440	9497	9555	9612	57
758	9669	9726	9784	9841	9890	9956	••13	••70	•127	•185	57
759	880242	0299	0356	0413	0471	0528	0585	0642	0699	0756	57
760	880814	0871	0928	0985	1042	1099	1156	1213	1271	1329	57
761	1385	1442	1499	1556	1613	1670	1727	1784	1841	1898	57
762	1955	2012	2069	2126	2183	2240	2297	2354	2411	2468	57
763	2525	2581	2638	2695	2752	2809	2866	2923	2980	3037	57
764	3093	3150	3207	3264	3321	3377	3434	3491	3548	3605	57
765	3661	3718	3775	3832	3888	3945	4002	4059	4115	4172	57
766	4229	4285	4342	4399	4455	4512	4569	4625	4682	4739	57
767	4795	4852	4909	4965	5022	5078	5135	5192	5248	5305	57
768	5361	5418	5474	5531	5587	5644	5700	5757	5813	5870	57
769	5926	5983	6039	6096	6152	6209	6265	6321	6378	6434	56
770	886491	6547	6604	6660	6716	6773	6829	6885	6948	6998	56
771	7054	7111	7167	7223	7280	7336	7392	7449	7505	7561	56
772	7617	7674	7730	7786	7842	7898	7955	8011	8067	8123	56
773	8179	8236	8292	8348	8404	8460	8516	8573	8629	8685	56
774	8741	8797	8853	8909	8965	9021	9077	9133	9190	9246	56
775	9302	9358	9414	9470	9526	9582	9638	9694	9750	9806	56
776	9862	9918	9974	••30	••86	•141	•197	•253	•309	•365	56
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779	1537	1593	1649	1705	1760	1816	1872	1928	1983	2039	56
780	892095	2150	2206	2262	2317	2373	2429	2484	2540	2595	56
781	2651	2707	2762	2818	2873	2929	2985	3040	3096	3151	56
782	3207	3262	3318	3373	3429	3484	3540	3595	3651	3706	56
783	3762	3817	3873	3928	3984	4039	4094	4150	4205	4261	55
784	4316	4371	4427	4482	4538	4593	4648	4704	4759	4814	55
785	4870	4925	4980	5036	5091	5146	5201	5257	5312	5367	55
786	5423	5478	5533	5588	5644	5699	5754	5809	5864	5920	55
787	5975	6030	6085	6140	6195	6251	6306	6361	6416	6471	55
788	6526	6581	6636	6692	6747	6802	6857	6912	6967	7022	55
789	7077	7132	7187	7242	7297	7352	7407	7462	7517	7572	55
790	897627	7682	7737	7792	7847	7902	7957	8012	8067	8122	55
791	8176	8231	8286	8341	8396	8451	8506	8561	8615	8670	55
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795	900367	0422	0476	0531	0586	0640	0695	0749	0804	0859	55
796	0913	0968	1022	1077	1131	1186	1240	1295	1349	1404	55
797	1458	1513	1567	1622	1676	1731	1785	1840	1894	1948	54
798	2003	2057	2112	2166	2221	2275	2329	2384	2438	2492	54
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802	4174	4229	4283	4337	4391	4445	4499	4553	4607	4661	54
803	4716	4770	4824	4878	4932	4986	5040	5094	5148	5202	54
804	5256	5310	5364	5418	5472	5526	5580	5634	5688	5742	54
805	5796	5850	5904	5958	6012	6066	6119	6173	6227	6281	54
806	6335	6389	6443	6497	6551	6604	6658	6712	6766	6820	54
807	6874	6927	6981	7035	7089	7143	7196	7250	7304	7358	54
808	7411	7465	7519	7573	7626	7680	7734	7787	7841	7895	54
809	7949	8002	8056	8110	8163	8217	8270	8324	8378	8431	54
810	908485	8539	8592	8646	8699	8753	8807	8860	8914	8967	54
811	9021	9074	9128	9181	9235	9289	9342	9396	9449	9503	54
812	9556	9610	9663	9716	9770	9823	9877	9930	9984	••37	53
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815	1158	1211	1264	1317	1371	1424	1477	1530	1584	1637	53
816	1690	1743	1797	1850	1903	1956	2009	2063	2116	2169	53
817	2222	2275	2328	2381	2435	2488	2541	2594	2647	2700	53
818	2753	2806	2859	2913	2966	3019	3072	3125	3178	3231	53
819	3284	3337	3390	3443	3496	3549	3602	3655	3708	3761	53
820	913814	3867	3920	3973	4026	4079	4132	4184	4237	4290	53
821	4343	4396	4449	4502	4555	4608	4660	4713	4766	4819	53
822	4872	4925	4977	5030	5083	5136	5189	5241	5294	5347	53
823	5400	5453	5505	5558	5611	5664	5716	5769	5822	5875	53
824	5927	5980	6033	6085	6138	6191	6243	6296	6349	6401	53
825	6454	6507	6559	6612	6664	6717	6770	6822	6875	6927	53
826	6980	7033	7085	7138	7190	7243	7295	7348	7400	7453	53
827	7506	7558	7611	7663	7716	7768	7820	7873	7925	7978	52
828	8030	8083	8135	8188	8240	8293	8345	8397	8450	8502	52
829	8555	8607	8659	8712	8764	8816	8869	8921	8973	9026	52
830	919078	9130	9183	9235	9287	9340	9392	9444	9496	9549	52
831	9601	9653	9706	9758	9810	9862	9914	9967	••19	••71	52
832	920123	0176	0228	0280	0332	0384	0436	0489	0541	0593	52
833	0645	0697	0749	0801	0853	0906	0958	1010	1062	1114	52
834	1166	1218	1270	1322	1374	1426	1478	1530	1582	1634	52
835	1686	1738	1790	1842	1894	1946	1998	2050	2102	2154	52
836	2206	2258	2310	2362	2414	2466	2518	2570	2622	2674	52
837	2725	2777	2829	2881	2933	2985	3037	3089	3140	3192	52
838	3244	3296	3348	3399	3451	3503	3555	3607	3658	3710	52
839	3762	3814	3865	3917	3969	4021	4072	4124	4176	4228	52
840	924279	4331	4383	4434	4486	4538	4589	4641	4693	4744	52
841	4796	4848	4899	4951	5003	5054	5106	5157	5209	5261	52
842	5312	5364	5415	5467	5518	5570	5621	5673	5725	5776	52
843	5828	5879	5931	5982	6034	6085	6137	6188	6240	6291	51
844	6342	6394	6445	6497	6548	6600	6651	6702	6754	6805	51
845	6857	6908	6959	7011	7062	7114	7165	7216	7268	7319	51
846	7370	7422	7473	7524	7576	7627	7678	7730	7781	7832	51
847	7843	7935	7986	8037	8088	8140	8191	8242	8293	8345	51
848	8396	8447	8498	8549	8601	8652	8703	8754	8805	8857	51
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853	0949	1000	1051	1102	1153	1204	1254	1305	1356	1407	51
854	1458	1509	1560	1610	1661	1712	1763	1814	1865	1915	51
855	1966	2017	2068	2118	2169	2220	2271	2322	2372	2423	51
856	2474	2524	2575	2626	2677	2727	2778	2829	2879	2930	51
857	2981	3031	3082	3133	3183	3234	3285	3335	3386	3437	51
858	3487	3538	3589	3639	3690	3740	3791	3841	3892	3943	51
859	3993	4044	4094	4145	4195	4246	4296	4347	4397	4448	51
860	934498	4549	4599	4650	4700	4751	4801	4852	4902	4953	50
861	5003	5054	5104	5154	5205	5255	5306	5356	5406	5457	50
862	5507	5558	5608	5658	5709	5759	5809	5860	5910	5960	50
863	6011	6061	6111	6162	6212	6262	6313	6363	6413	6463	50
864	6514	6564	6614	6665	6715	6765	6815	6865	6916	6966	50
865	7016	7066	7117	7167	7217	7267	7317	7367	7418	7468	50
866	7518	7568	7618	7668	7718	7769	7819	7869	7919	7969	50
867	8019	8069	8119	8169	8219	8269	8320	8370	8420	8470	50
868	8520	8570	8620	8670	8720	8770	8820	8870	8920	8970	50
869	9020	9070	9120	9170	9220	9270	9320	9369	9419	9469	50
870	939519	9569	9619	9669	9719	9769	9819	9869	9918	9968	50
871	940018	0068	0118	0168	0218	0267	0317	0367	0417	0467	50
872	0515	0566	0616	0666	0716	0765	0815	0865	0915	0964	50
873	1014	1064	1114	1163	1213	1263	1313	1362	1412	1462	50
874	1511	1561	1611	1660	1710	1760	1809	1859	1909	1958	50
875	2008	2058	2107	2157	2207	2256	2306	2355	2405	2455	50
876	2504	2554	2603	2653	2702	2752	2801	2851	2901	2950	50
877	3000	3049	3099	3148	3198	3247	3297	3346	3396	3445	49
878	3495	3544	3593	3643	3692	3742	3791	3841	3890	3939	49
879	3989	4038	4088	4137	4186	4236	4285	4335	4384	4433	49
880	944483	4532	4581	4631	4680	4729	4779	4828	4877	4927	49
881	4976	5025	5074	5124	5173	5222	5272	5321	5370	5419	49
882	5469	5518	5567	5616	5665	5715	5764	5813	5862	5912	49
883	5961	6010	6059	6108	6157	6207	6256	6305	6354	6403	49
884	6452	6501	6551	6600	6649	6698	6747	6796	6845	6894	49
885	6943	6992	7041	7090	7140	7189	7238	7287	7336	7385	49
886	7434	7483	7532	7581	7630	7679	7728	7777	7826	7875	49
887	7924	7973	8022	8070	8119	8168	8217	8266	8315	8364	49
888	8413	8462	8511	8560	8609	8657	8706	8755	8804	8853	49
889	8902	8951	8999	9048	9097	9146	9195	9244	9292	9341	49
890	949390	9439	9488	9536	9585	9634	9683	9731	9780	9829	49
891	9878	9926	9975	••24	••73	•121	•170	•219	•267	•316	49
892	950365	0414	0462	0511	0560	0608	0657	0706	0754	0803	49
893	0851	0900	0949	0997	1046	1095	1143	1192	1240	1289	49
894	1338	1386	1435	1483	1532	1580	1629	1677	1726	1775	49
895	1823	1872	1920	1969	2017	2066	2114	2163	2211	2260	48
896	2308	2356	2405	2453	2502	2550	2599	2647	2696	2744	48
897	2792	2841	2889	2938	2986	3034	3083	3131	3180	3228	48
898	3276	3325	3373	3421	3470	3518	3566	3615	3663	3711	48
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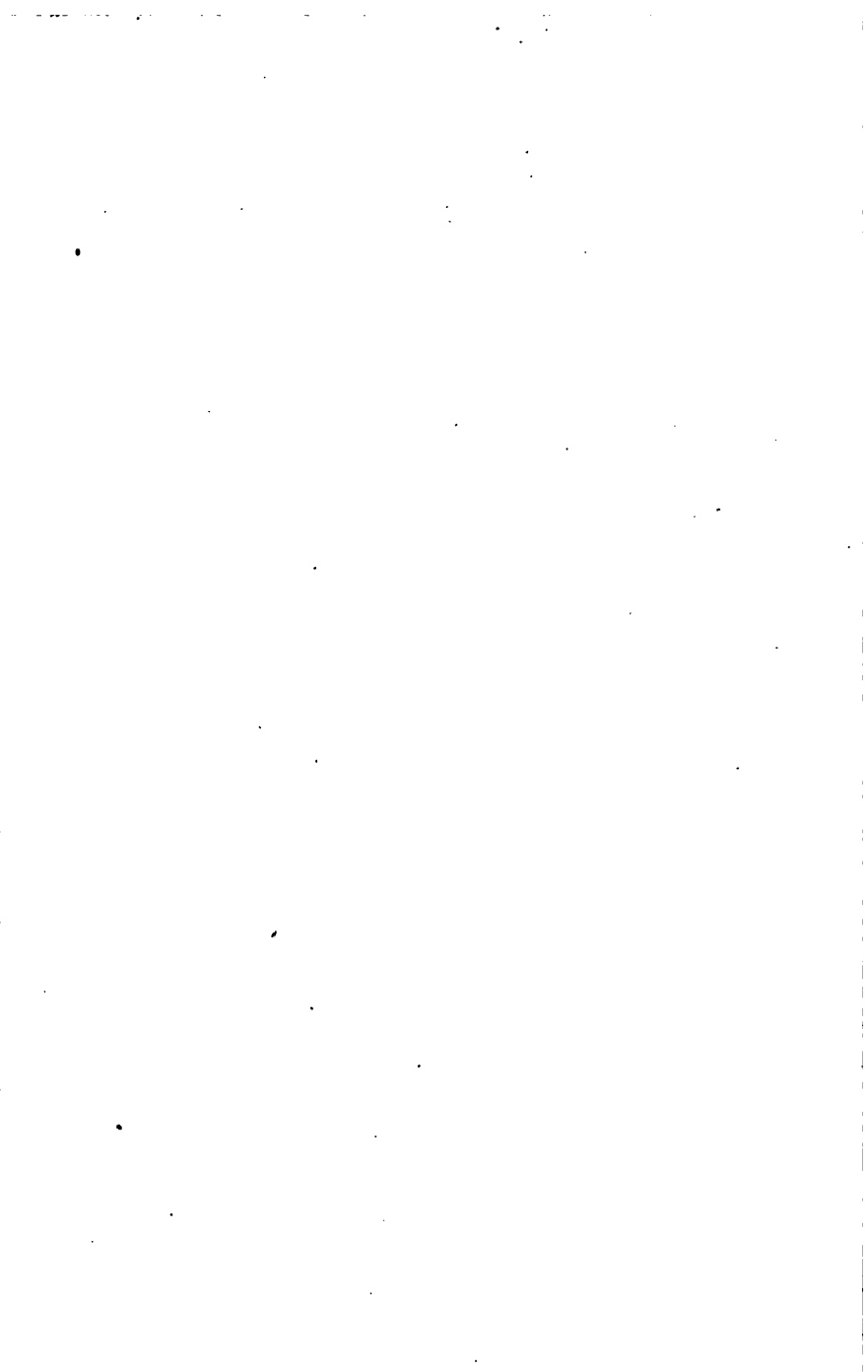
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903	5688	5736	5784	5832	5880	5928	5976	6024	6072	6120	48
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906	7128	7176	7224	7272	7320	7368	7416	7464	7512	7559	48
907	7607	7655	7703	7751	7799	7847	7894	7942	7990	8038	48
908	8086	8134	8181	8229	8277	8325	8373	8421	8468	8516	48
909	8564	8612	8659	8707	8755	8803	8850	8898	8946	8994	48
910	959041	9089	9137	9185	9232	9280	9328	9375	9423	9471	48
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912	9995	*42	*90	*138	*185	*233	*280	*328	*376	*423	48
913	960471	0518	0566	0613	0661	0709	0756	0804	0851	0899	48
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915	1421	1469	1516	1563	1611	1658	1706	1753	1801	1848	47
916	1895	1943	1990	2038	2085	2132	2180	2227	2275	2322	47
917	2369	2417	2464	2511	2559	2606	2653	2701	2748	2795	47
918	2843	2890	2937	2985	3032	3079	3126	3174	3221	3268	47
919	3316	3363	3410	3457	3504	3552	3599	3646	3693	3741	47
920	963788	3835	3882	3929	3977	4024	4071	4118	4165	4212	47
921	4260	4307	4354	4401	4448	4495	4542	4590	4637	4684	47
922	4731	4778	4825	4872	4919	4966	5013	5061	5108	5155	47
923	5202	5249	5296	5343	5390	5437	5484	5531	5578	5625	47
924	5672	5719	5766	5813	5860	5907	5954	6001	6048	6095	47
925	6142	6189	6236	6283	6329	6376	6423	6470	6517	6564	47
926	6611	6658	6705	6752	6799	6845	6892	6939	6986	7033	47
927	7080	7127	7173	7220	7267	7314	7361	7408	7454	7501	47
928	7548	7595	7642	7688	7735	7782	7829	7875	7922	7969	47
929	8016	8062	8109	8156	8203	8249	8296	8343	8390	8436	47
930	968483	8530	8576	8623	8670	8716	8763	8810	8856	8903	47
931	8950	8996	9043	9090	9136	9183	9229	9276	9323	9369	47
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933	9882	9928	9975	*21	*68	*114	*161	*207	*254	*300	47
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935	0812	0858	0904	0951	0997	1044	1090	1137	1183	1229	46
936	1276	1322	1369	1415	1461	1508	1554	1601	1647	1693	46
937	1740	1786	1832	1879	1925	1971	2018	2064	2110	2157	46
938	2203	2249	2295	2342	2388	2434	2481	2527	2573	2619	46
939	2666	2712	2758	2804	2851	2897	2943	2989	3035	3082	46
940	973128	3174	3220	3266	3313	3359	3405	3451	3497	3543	46
941	3590	3636	3682	3728	3774	3820	3866	3913	3959	4005	46
942	4051	4097	4143	4189	4235	4281	4327	4374	4420	4466	46
943	4512	4558	4604	4650	4696	4742	4788	4834	4880	4926	46
944	4972	5018	5064	5110	5156	5202	5248	5294	5340	5386	46
945	5432	5478	5524	5570	5616	5662	5707	5753	5799	5845	46
946	5891	5937	5983	6029	6075	6121	6167	6213	6258	6304	46
947	6350	6396	6442	6488	6533	6579	6625	6671	6717	6763	46
948	6808	6854	6900	6946	6992	7037	7083	7129	7175	7220	46
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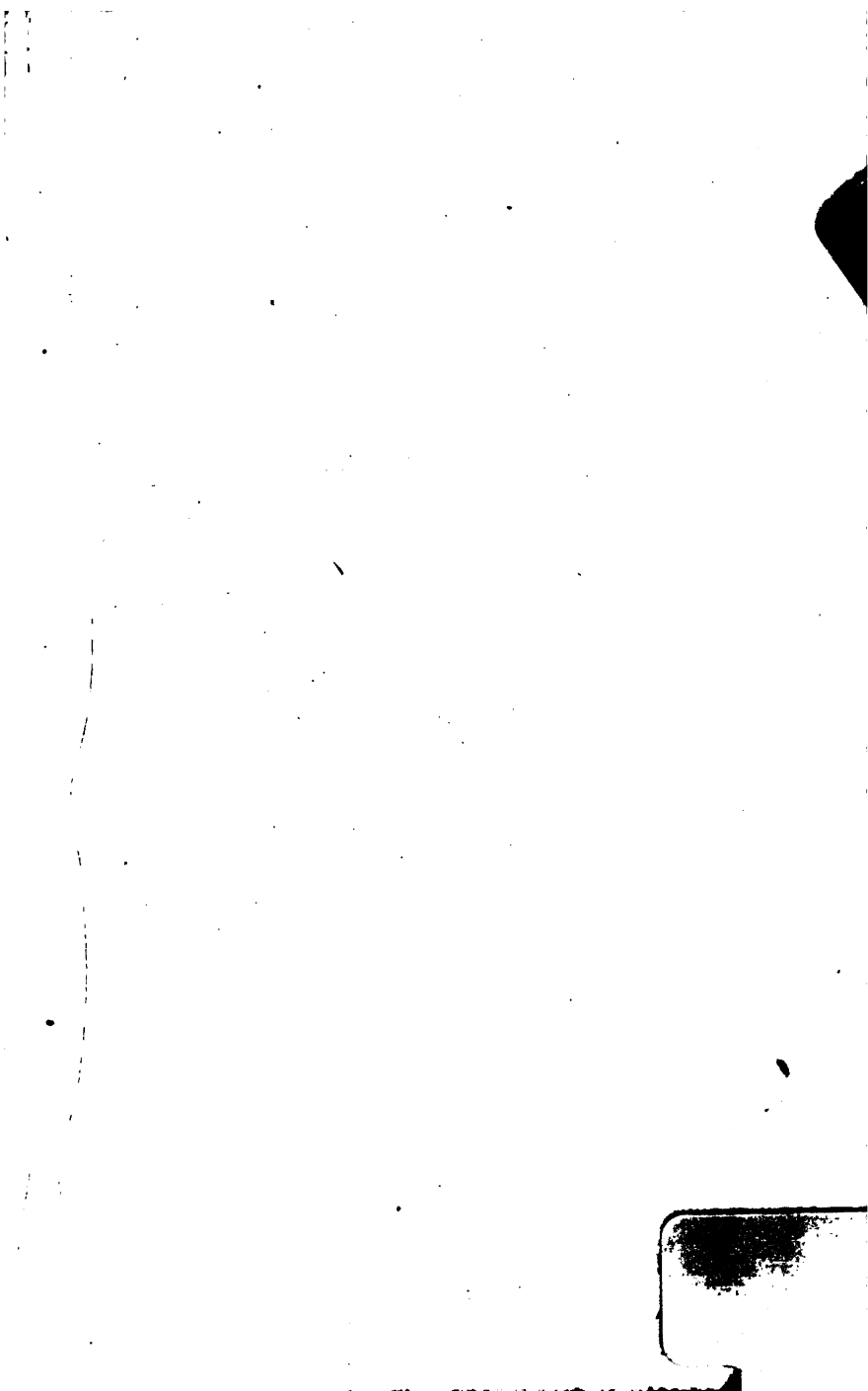
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952	8637	8683	8728	8774	8819	8865	8911	8956	9002	9047	46
953	9093	9138	9184	9230	9275	9321	9366	9412	9457	9503	46
954	9548	9594	9639	9685	9730	9776	9821	9867	9912	9958	46
955	980003	0049	0094	0140	0185	0231	0276	0322	0367	0412	45
956	0458	0503	0549	0594	0640	0685	0730	0776	0821	0867	45
957	0912	0957	1003	1048	1093	1139	1184	1229	1275	1320	45
958	1366	1411	1456	1501	1547	1592	1637	1683	1728	1773	45
959	1819	1864	1909	1954	2000	2045	2090	2135	2181	2226	45
960	982271	2316	2362	2407	2452	2497	2543	2588	2633	2678	45
961	2723	2769	2814	2859	2904	2949	2994	3040	3085	3130	45
962	3175	3220	3265	3310	3356	3401	3446	3491	3536	3581	45
963	3626	3671	3716	3762	3807	3852	3897	3942	3987	4032	45
964	4077	4122	4167	4212	4257	4302	4347	4392	4437	4482	45
965	4527	4572	4617	4662	4707	4752	4797	4842	4887	4932	45
966	4977	5022	5067	5112	5157	5202	5247	5292	5337	5382	45
967	5426	5471	5516	5561	5606	5651	5696	5741	5786	5830	45
968	5875	5920	5965	6010	6055	6100	6144	6189	6234	6279	45
969	6324	6369	6413	6458	6503	6548	6593	6637	6682	6727	45
970	986772	6817	6861	6906	6951	6996	7040	7085	7130	7175	45
971	7219	7264	7309	7353	7398	7443	7488	7532	7577	7622	45
972	7666	7711	7756	7800	7845	7890	7934	7979	8024	8068	45
973	8113	8157	8202	8247	8291	8336	8381	8425	8470	8514	45
974	8559	8604	8648	8693	8737	8782	8826	8871	8916	8960	45
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978	990339	0383	0428	0472	0516	0561	0605	0650	0694	0738	44
979	0783	0827	0871	0916	0960	1004	1049	1093	1137	1182	44
980	991226	1270	1315	1359	1403	1448	1492	1536	1580	1625	44
981	1669	1713	1758	1802	1846	1890	1935	1979	2023	2067	44
982	2111	2156	2200	2244	2288	2333	2377	2421	2465	2509	44
983	2554	2598	2642	2686	2730	2774	2819	2863	2907	2951	44
984	2995	3039	3083	3127	3172	3216	3260	3304	3348	3392	44
985	3436	3480	3524	3568	3613	3657	3701	3745	3789	3833	44
986	3877	3921	3965	4009	4053	4097	4141	4185	4229	4273	44
987	4317	4361	4405	4449	4493	4537	4581	4625	4669	4713	44
988	4757	4801	4845	4889	4933	4977	5021	5065	5108	5152	44
989	5196	5240	5284	5328	5372	5416	5460	5504	5547	5591	44
990	995635	5679	5723	5767	5811	5854	5898	5942	5986	6030	44
991	6074	6117	6161	6205	6249	6293	6337	6380	6424	6468	44
992	6512	6555	6599	6643	6687	6731	6774	6818	6862	6906	44
993	6949	6993	7037	7080	7124	7168	7212	7255	7299	7343	44
994	7386	7430	7474	7517	7561	7605	7648	7692	7736	7779	44
995	7823	7867	7910	7954	7998	8041	8085	8129	8172	8216	44
996	8259	8303	8347	8390	8434	8477	8521	8564	8608	8652	44
997	8695	8739	8782	8826	8869	8913	8956	9000	9043	9087	44
998	9131	9174	9218	9261	9305	9348	9392	9435	9479	9522	44
999	9565	9609	9652	9696	9739	9783	9826	9870	9913	9957	43
N.	0	1	2	3	4	5	6	7	8	9	D.







A TABLE OF LOGARITHMS FROM 1 TO 10,000.

N.	0	1	2	3	4	5	6	7	8	9	D.
100	000000	0434	0868	1301	1734	2166	2598	3029	3461	3891	432
101	4321	4751	5181	5609	6038	6466	6894	7321	7748	8174	428
102	8600	9026	9451	9876	*300	*724	1147	1570	1993	2415	424
103	012837	3259	3680	4100	4521	4940	5360	5779	6197	6616	419
104	7033	7451	7868	8284	8700	9116	9532	9947	*361	*775	416
105	021189	1603	2016	2428	2841	3252	3664	4075	4486	4896	412
106	5306	5715	6125	6533	6942	7350	7757	8164	8571	8978	408
107	9384	9789	*195	*600	1004	1408	1812	2216	2619	3021	404
108	033424	3826	4227	4628	5029	5430	5830	6230	6629	7028	400
109	7426	7825	8223	8620	9017	9414	9811	*207	*602	*998	396
110	041393	1787	2182	2576	2969	3362	3755	4148	4540	4932	393
111	5323	5714	6105	6495	6885	7275	7664	8053	8442	8830	359
112	9218	9606	9993	*380	*766	1153	1538	1924	2309	2694	386
113	053078	3463	3846	4230	4613	4996	5378	5760	6142	6524	382
114	6905	7286	7666	8046	8426	8805	9185	9563	9942	*320	379
115	060698	1075	1452	1829	2206	2582	2958	3333	3709	4083	376
116	4458	4832	5206	5580	5953	6326	6699	7071	7443	7815	372
117	8186	8557	8928	9298	9668	*38	*407	*776	1145	1514	369
118	071882	2250	2617	2985	3352	3718	4085	4451	4816	5182	366
119	5547	5912	6276	6640	7004	7368	7731	8094	8457	8819	363
120	079181	9543	9904	*266	*626	*987	1347	1707	2067	2426	360
121	082785	3144	3503	3861	4219	4576	4934	5291	5647	6004	357
122	6360	6716	7071	7426	7781	8136	8490	8845	9198	9552	355
123	9905	*258	*611	*963	1315	1667	2018	2370	2721	3071	351
124	093422	3772	4122	4471	4820	5169	5518	5866	6215	6562	349
125	6910	7257	7604	7951	8298	8644	8990	9335	9681	*226	346
126	100371	0715	1059	1403	1747	2091	2434	2777	3119	3462	343
127	3804	4146	4487	4828	5169	5510	5851	6191	6531	6871	340
128	7210	7549	7888	8227	8565	8903	9241	9579	9916	*253	338
129	110590	0926	1263	1599	1934	2270	2605	2940	3275	3609	335
130	113943	4277	4611	4944	5278	5611	5943	6276	6608	6940	333
131	7271	7603	7934	8265	8595	8926	9256	9586	9915	*245	330
132	120574	0903	1231	1560	1888	2216	2544	2871	3198	3525	328
133	3852	4178	4504	4830	5156	5481	5806	6131	6456	6781	325
134	7105	7429	7753	8076	8399	8722	9045	9368	9690	*12	323
135	130334	0655	0977	1298	1619	1939	2260	2580	2900	3219	321
136	3539	3858	4177	4496	4814	5133	5451	5769	6086	6403	318
137	6721	7037	7354	7671	7987	8303	8618	8934	9249	9564	315
138	9879	*194	*508	*822	1136	1450	1763	2076	2389	2702	314
139	143015	3327	3639	3951	4263	4574	4885	5196	5507	5818	311
140	146128	6438	6748	7058	7367	7676	7985	8294	8603	8911	309
141	9219	9527	9835	*142	*449	*756	1063	1370	1676	1982	307
142	152288	2594	2900	3205	3510	3815	4120	4424	4728	5032	305
143	5336	5640	5943	6246	6549	6852	7154	7457	7759	8061	303
144	8362	8664	8965	9266	9567	9868	*168	*469	*769	1068	301
145	161368	1667	1967	2266	2564	2863	3161	3460	3758	4055	299
146	4353	4650	4947	5244	5541	5838	6134	6430	6726	7022	297
147	7317	7613	7908	8203	8497	8792	9086	9380	9674	9968	295
148	170262	0555	0848	1141	1434	1726	2019	2311	2603	2895	293
149	3186	3478	3769	4060	4351	4641	4932	5222	5512	5802	291
N.	0	1	2	3	4	5	6	7	8	9	D.

A TABLE OF LOGARITHMS FROM 1 TO 10,000.

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N.	0	1	2	3	4	5	6	7	8	9	D.
150	176091	6381	6670	6959	7248	7536	7825	8113	8401	8689	289
151	8977	9264	9552	9839	*126	*413	*699	*985	1272	1558	287
152	181844	2129	2415	2700	2985	3270	3555	3839	4123	4407	285
153	4691	4975	5259	5542	5825	6108	6391	6674	6956	7239	283
154	7521	7803	8084	8366	8647	8928	9209	9490	9771	*51	281
155	190332	0612	0892	1171	1451	1730	2010	2289	2567	2846	279
156	3125	3403	3681	3959	4237	4514	4792	5069	5346	5623	278
157	5900	6176	6453	6729	7005	7281	7556	7832	8107	8382	276
158	8657	8932	9206	9481	9755	*29	*303	*577	*850	1124	274
159	201397	1670	1943	2216	2488	2761	3033	3305	3577	3848	272
160	204120	4391	4663	4934	5204	5475	5746	6016	6286	6556	271
161	6826	7096	7365	7634	7904	8173	8441	8710	8979	9247	269
162	9515	9783	*51	*319	*586	*853	1121	1388	1654	1921	267
163	212188	2454	2720	2986	3252	3518	3783	4049	4314	4579	266
164	4844	5109	5373	5638	5902	6166	6430	6694	6957	7221	264
165	7484	7747	8010	8273	8536	8798	9060	9323	9585	9846	262
166	220108	0370	0631	0892	1153	1414	1675	1936	2196	2456	261
167	2716	2976	3236	3496	3755	4015	4274	4533	4792	5051	259
168	5309	5568	5826	6084	6342	6600	6858	7115	7372	7630	258
169	7887	8144	8400	8657	8913	9170	9426	9682	9938	*193	256
170	230449	0704	0960	1215	1470	1724	1979	2234	2488	2742	254
171	2996	3250	3504	3757	4011	4264	4517	4770	5023	5276	253
172	5528	5781	6033	6285	6537	6789	7041	7292	7544	7795	252
173	8046	8297	8548	8799	9049	9299	9550	9800	*50	*300	250
174	240549	0799	1048	1297	1546	1795	2044	2293	2541	2790	249
175	3038	3286	3534	3782	4030	4277	4525	4772	5019	5266	248
176	5513	5759	6006	6252	6499	6745	6991	7237	7482	7728	246
177	7973	8219	8464	8709	8954	9198	9443	9687	9932	*176	245
178	250420	0664	0908	1151	1395	1638	1881	2125	2368	2610	243
179	2853	3096	3338	3580	3822	4064	4306	4548	4790	5031	242
180	255273	5514	5755	5996	6237	6477	6718	6958	7198	7439	241
181	7679	7918	8158	8398	8637	8877	9116	9355	9594	9833	239
182	260071	0310	0548	0787	1025	1263	1501	1739	1976	2214	238
183	2451	2688	2925	3162	3399	3636	3873	4109	4346	4582	237
184	4818	5054	5290	5525	5761	5996	632	6467	6702	6937	235
185	7172	7406	7641	7875	8110	8344	8578	8812	9046	9279	234
186	9513	9746	9980	*213	*446	*679	*912	1144	1377	1609	233
187	271842	2074	2306	2538	2770	3001	3233	3464	3696	3927	232
188	4153	4389	4620	4850	5081	5311	5542	5772	6002	6232	230
189	6462	6692	6921	7151	7380	7609	7838	8067	8296	8525	229
190	278754	8982	9211	9439	9667	9895	*123	*351	*578	*806	228
191	281033	1261	1488	1715	1942	2169	2396	2622	2849	3075	227
192	3301	3527	3753	3979	4205	4431	4656	4882	5107	5332	226
193	5557	5782	6007	6232	6456	6681	6905	7130	7354	7578	225
194	7802	8026	8249	8473	8696	8920	9143	9366	9589	9812	223
195	290035	0257	0480	0702	0925	1147	1369	1591	1813	2034	222
196	2256	2478	2699	2920	3141	3363	3584	3804	4025	4246	221
197	4466	4687	4907	5127	5347	5567	5787	6007	6226	6446	220
198	6665	6884	7104	7323	7542	7761	7979	8198	8416	8635	219
199	8853	9071	9289	9507	9725	9943	*161	*378	*595	*813	218
N.	0	1	2	3	4	5	6	7	8	9	D.

